Identification and Inference in First-Price Auctions with Risk Averse Bidders and Selective Entry*

Xiaohong Chen† Matthew Gentry‡ Tong Li§ Jingfeng Lu¶

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Abstract

We study identification and inference in first-price auctions with risk averse bidders and selective entry, building on a flexible framework we call the Affiliated Signal with Risk Aversion (AS-RA) model. Assuming exogenous variation in either the number of potential bidders (N) or a continuous instrument (z) shifting opportunity costs of entry, we provide a sharp characterization of the nonparametric restrictions implied by equilibrium bidding. This characterization implies that risk neutrality is nonparametrically testable. In addition, with sufficient variation in both N and z, the AS-RA model primitives are nonparametrically identified (up to a bounded constant) on their equilibrium domains. Finally, we explore new methods for inference in set-identified auction models based on Chen, Christensen, and Tamer (2018), as well as novel and fast computational strategies using Mathematical Programming with Equilibrium Constraints. Simulation studies reveal good finite-sample performance of our inference methods, which can readily be adapted to other set-identified flexible equilibrium models with parameter dependent support.

Keywords: Auctions, entry, risk aversion, boundary condition, identification, set inference, parameter-dependent support, MPEC, flexible parametric form, approximate profile likelihood-ratio, Bayes credible sets, frequentist confidence sets.

JEL Classifications: D44, C57.

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†Xiaohong Chen: Department of Economics, Yale University. Email: xiaohong.chen@yale.edu.

‡Matthew Gentry: Department of Economics, Florida State University. Email: mgentry@fsu.edu.

§Tong Li: Department of Economics, Vanderbilt University. Email: tong.li@vanderbilt.edu.

¶Jingfeng Lu: Department of Economics, National University of Singapore. Email: ecsljf@nus.edu.sg.

1 Introduction

Risk aversion and entry are both important considerations in real-world auction markets. Unfortunately, the interaction between these factors also raises a significant empirical challenge: selection into entry may undermine the exclusion restrictions necessary for identification of risk preferences. This paper provides a comprehensive analysis of identification in first price auctions with risk averse bidders and selective entry, allowing for potentially non-binding reserve prices. We then explore inference based on our identification results, applying and extending some results of Chen, Christensen, and Tamer (2018) (henceforth CCT) to construct valid confidence sets for identified sets for potentially set-identified model primitives.

Bidder risk attitudes are of fundamental importance in auction design—affecting, among other things, the revenue ranking between first-price and ascending auctions (Maskin and Riley (1984)), the structure of the optimal mechanism (Matthews (1987)), and whether the seller should disclose reserve prices (Li and Tan (2017)). Motivated by this fact, a substantial empirical literature has arisen on bidder risk preferences, finding evidence for risk aversion in a variety of real-world contexts, including in settings where bidders are firms. For instance, Athey and Levin (2001) find that bidding firms diversify risk across species in U.S. Forest Service timber auctions; Ackerberg, Hirano, and Shahriar (2017) show that bidder risk aversion rationalizes the use of buy-it-now options in eBay auctions; and Bajari and Hortacsu (2005) find that risk aversion explains bidder behavior in experiments. Using structural approaches, Lu and Perrigne (2008) and Campo, Guerre, Perrigne, and Vuong (2011) (henceforth CGPV) find evidence for risk aversion in U.S. Forest Service timber auctions, while Kong (2020) finds that risk aversion can explain observed revenue differences between first-price and ascending auctions for oil and gas leases in New Mexico.[1]

A similarly substantial body of empirical research has also documented the prevalence of endogenous entry in real-world auction markets,[2] which has also motivated theoretical work on selective entry in auction models.[3] While this literature has evolved largely in parallel to

[1]Findings of risk aversion in timber, oil and gas auctions are of particular interest as the players in both markets are firms. We view such findings as consistent with the hypothesis that, even within firms, all bidding is ultimately done by individuals. Hence, as usual in principle-agent models, the risk preferences of the bidding agents will typically be relevant even if one presumes that the firm itself is risk neutral.

[2]For instance, Hendricks, Pinkse, and Porter (2003) report that less than 25 percent of eligible bidders actually participate in U.S. Minerals Management Service "wildcat auctions" held from 1954 to 1970. Li and Zheng (2009) find that only about 28 percent of planholders in Texas Department of Transportation mowing contracts actually submit bids. Similar patterns have been reported for timber auctions (Athey, Levin, and Seira (2011), Li and Zhang (2010), Roberts and Sweeting (2013)), online auctions (Bajari and Hortacsu (2003)), highway procurement (Krasnokutskaya and Seim (2011)) and corporate takeover markets (Gentry and Stroup (2019)) among others.

[3]See, for example, Samuelson (1985), Ye (2007), Roberts and Sweeting (2013), Marmer, Shneyerov, and Xu (2013), Gentry and Li (2014) and references therein.

the literature on risk aversion, the conjunction between entry and risk aversion raises new, important economic questions. For instance, Smith and Levin (1996) show that in settings with both risk aversion and entry, second-price auctions can yield higher revenue than first-price auctions, contradicting the usual revenue ranking under risk aversion (Maskin and Riley (1984)). Answers to many other counterfactual questions of policy interest—such as what reserve price maximizes revenue, whether to disclose the number of bidders, how revenue varies with the number of potential bidders, and how sellers should regulate participation—will similarly depend on the interaction between risk aversion and entry.

Econometrically, however, the interaction between risk aversion and entry also raises substantial challenges for identification and inference, particularly in settings where entry is potentially selective. Existing results on nonparametric point identification in auctions with risk averse bidders assume that the latent distribution of bidder valuations is invariant either to the seller's choice of auction format (Lu and Perrigne (2008)), or to the set of competitors faced (Guerre, Perrigne, and Vuong (2009), henceforth GPV). But, as shown by Li, Lu, and Zhao (2015) (henceforth LLZ), if risk averse bidders select into entry, the distribution of valuations among entrants will respond endogenously to both the auction format and the set of competitors faced. Hence both invariance assumptions typically fail in settings with selective entry, rendering identification of risk preferences—and therefore of any counterfactual whose answer depends on risk preferences—correspondingly uncertain.

Motivated by these observations, we study identification and inference in first-price auctions with risk averse bidders and selective entry, building on a flexible framework we label the Affiliated Signal with Risk Aversion (AS-RA) model. This model considers N symmetric potential bidders with wealth preferences described by a smooth concave Bernoulli utility function U, who compete in a first-price auction with entry. Potential bidders have independent private values, observe signals of their values prior to entry, and choose whether to incur a common-knowledge entry cost, with entrants learning their values and submitting bids subject to a weakly non-binding reserve price. The AS-RA framework nests many existing models as special cases, including the affiliated-signal models of Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2014) (henceforth GL); the mixed-strategy entry model of Levin and Smith (1994); and the models of risk averse bidders with exogenous entry in GPV and CGPV. It therefore provides a natural focal point for researchers seeking to understand the policy implications of interactions between risk aversion and selective entry.

Our paper makes three main contributions. **First**, we generalize the results of LLZ on existence, uniqueness, and properties of symmetric monotone equilibrium in the AS-RA model to allow for potentially non-binding reserve prices (Theorem 4 in Appendix A). As pointed out by a referee, this generalization is very important empirically, since reserve prices may or may not bind in practice. It is also novel theoretically, since it changes the boundary conditions for equilibrium bidding: whereas with full participation the lowest-type bidder will bid their value, when there is a positive probability that no rival enters, the lowest-type bidder will bid the reserve price, which may be strictly below their value. If so, this introduces a discontinuity in the minimum bid as rivals transition from partial to complete entry. We show that equilibrium payoffs must nevertheless be pointwise continuous in rival entry probabilities for all but the lowest bidder type. This in turn suffices to guarantee existence of a symmetric monotone equilibrium, which also must be unique.

Second, we establish new results on nonparametric and semiparametric identification in first-price auctions with both risk averse bidders and selective entry. Following GL, we consider identification based on variation either in the number of potential competitors N or in an instrument z influencing bidders' opportunity costs of entry. Assuming that neither potential bidders' utility U nor the *ex ante* distribution of bidders' private information depends on realizations of N and z, we provide a *sharp* characterization of the set of AS-RA primitives consistent with equilibrium bidding behavior (Theorem 1). We show that risk neutrality is nonparametrically testable even with variation in N only (Corollary 1). More generally, given sufficient variation in both N and z, model primitives are nonparametrically identified up to a bounded constant (Theorem 2). In the process, we show that the difference in boundary conditions noted above substantially changes the identification problem, requiring our novel identification results even when (for example) entry is non-selective. Finally, we explore semiparametric identification within the AS-RA model, showing that the CRRA and CARA utility families imply semiparametric point identification of U (Corollary 3), and that a parametric value-signal copula family yields conditional identification of model primitives as functions of the copula parameters (Theorem 3).

Third, we adapt and extend the quasi-Bayes log-likelihood-ratio (LR) approach of CCT to construct frequentist confidence sets (CSs) for the identified sets of AS-RA model primitives. Likelihood based inference on (possibly) partially identified first-price auction models like ours is particularly challenging because (i) the support of equilibrium bids is parameter dependent; and (ii) the log-likelihood functions typically do not have closed-form expressions. Problem (i) violates usual regularity conditions for the asymptotic normality of maximum-likelihood estimator (MLE) even in point-identified parametric auction models; see, e.g., Donald and Paarsch (1993), Hirano and Porter (2003) and Chernozhukov and Hong (2004). Luckily, CCT show in their Appendix C that their Procedure 1, which provides quasi-Bayes LR based CSs for the identified sets in set-identified models, yields valid frequentist CSs even when the support of observables depends on parameters. We further show in our Appendix B that CCT's Procedure 2, which constructs quasi-Bayes profiled LR based CSs for the identified sets for subvector parameters in set-identified models, also yields valid frequentist CSs under parameter-dependent support. This result is of independent interest. To address Problem (ii), we provide computationally efficient algorithms for implementing quasi-Bayes profiled LR CSs in potentially set identified auction models without closed-form likelihood expressions. We parameterize bidder utility and distributions of values among entrants flexibly using Bernstein polynomial bases, and develop CSs for identified sets of parameters and of parameter subvectors. We propose novel and fast algorithms for implementation, building on Mathematical Programming with Equilibrium Constraints (MPEC) (see Su and Judd (2012)) and Hamiltonian Monte Carlos. To our knowledge, our paper is the first to adapt CCT's Procedures 1 and 2 to the challenging problem of set inference in auction models. Results from large-scale, realistically complex simulations indicate that ours is a very promising approach to inference problems in set identified equilibrium models with parameter dependent support. Moreover, simulations show that, even for point-identified AS-RA models, our CSs for a scalar parameter (e.g., a risk-attitude parameter) perform as well as, or even better than, standard profiled LR CSs and percentile CSs for models.

The rest of the paper is organized as follows. Section 2 introduces the AS-RA model with general boundary conditions, with further details in Appendix A. Section 3 characterizes identified sets for model primitives. Section 4 presents quasi-Bayes LR based confidence sets for model primitives, with further details in Appendix B. Section 5 conducts a non-trivial simulation study in which bidders' risk preferences and value distributions are approximated by Bernstein polynomials. Online Appendix **??** presents computation details; and online Appendix **??** contains mathematical proofs of all the technical results.

2 The symmetric AS-RA model

We consider a population of independent first-price auctions, each involving allocation of a single indivisible good among $N\,(\geq 2)$ potential bidders via a first-price auction with entry. The number of potential bidders N varies on the set $\mathcal{N} \equiv \{N_1, ..., N_K\}$, where elements are ordered such that $N_1 < N_2 < ... < N_K$, and the subscript $k \in \mathcal{K} \equiv \{1, ..., K\}$ indexes levels of N. For each auction, the econometrician observes the number of potential bidders N, the number of bidders (entrants) n, the vector of submitted bids $\mathbf{b}$, and may also observe an entry instrument $z \in \mathcal{Z}$. We allow $\mathcal{Z}$ to be a singleton (corresponding to the case of no instrument), a discrete finite set, or a closed interval with non-empty interior. We focus on a symmetric environment with independent private information, although our main identification insights extend to asymmetric bidders and unobserved auction heterogeneity as in GL. Our identification results also extend immediately conditional on further auction covariates

X, although for simplicity we suppress these in notation.

2.1 Model overview

We model entry and bidding as a two-stage game with the following timing. First, in Stage 1, each potential bidder i receives a private signal S_i of her (unknown) private value V_i, and all potential bidders simultaneously decide whether to enter at an opportunity cost $c(z) > 0$, which potentially depends on an instrument $z \in \mathcal{Z}$ affecting the value of opportunities foregone by entry. Then, in Stage 2, the n bidders who choose to enter in Stage 1 learn the realizations v_i of their private values V_i and submit bids. The Stage 2 mechanism is a first-price auction with reserve price p_0, where the highest bidder wins and pays her bid.

Value-signal pairs (V_i, S_i) are drawn independently across bidders from a common joint distribution $F_{vs}(v, s)$, where the distribution of V_i given S_i is at least weakly stochastically increasing in S_i. We assume that V_i has a continuous marginal distribution F with support $[\underline{v}, \bar{v}]$, where $\underline{v} \geq 0$ and $\bar{v} \in (\underline{v}, \infty)$. We focus on settings where the reserve price p_0 is weakly non-binding, in the sense that $p_0 \leq \underline{v}$; a binding reserve price $p_0 > \underline{v}$ would induce well-known truncation issues which are not our primary interest in this paper. Finally, without loss of generality, we normalize Stage 1 signals to standard uniform: $S_i \sim U[0, 1]$. By Sklar's theorem (see, e.g., Nelsen (1999)), we then have $F_{vs}(v, s) = C(F(v), s)$, where $C(a, s)$ is the unique bivariate copula describing dependence between V_i and S_i.

Potential bidders are risk averse with risk preferences described by a symmetric, strictly monotone, weakly concave Bernoulli utility function $U(w)$, where w is post-auction wealth. Without loss of generality, we normalize U such that $U(0) = 0$ and $U(1) = 1$. For simplicity, we model bidders as having zero initial wealth and zero financial costs of entry, with $c(z)$ interpreted as a pure opportunity cost of entry following Lu (2009). As described in Appendix A, however, these are in fact equivalent to normalizations in a more general setting with nonzero initial wealth and both financial and opportunity costs of entry.

The number of potential competitors N, entry cost $c(z)$, reserve price p_0, utility function U, ex ante value distribution F, and value-signal copula C are known to all potential bidders, with value-signal realization (v_i, s_i) being private information revealed to potential bidder i with timing described above. Although N is common knowledge prior to entry, the number of entrants (actual bidders) n is revealed to bidders only after the auction concludes. In our view, this informational structure best reflects institutional practices typical in sealed-bid markets, where auctioneer announcements or industry experience convey knowledge of potential competition but bids are revealed only after the auction concludes.[4]

[4]For example, in US highway procurement markets, the auctioneer will typically publish a list of planhold-

2.2 Structural assumptions

In what follows, we refer to (U, F, C, c) as the model primitives, and (U, F, C) as the bid-stage primitives. We shall study identification of bid-stage primitives based on variation in N and / or z, assuming that both factors are excludable in the sense that true bid-stage primitives, subsequently denoted (U_0, F_0, C_0), are invariant to realizations of N and z.

Assumption 1. (U_0, F_0, C_0) and $c(\cdot)$ satisfy the following conditions:

1. For all $N \in \mathcal{N}$ and $z \in \mathcal{Z}$, $Pr(V_i \leq v|N, z) = F_0(v)$ for any $v \in [0, \infty)$ and $Pr(F_0(V_i) \leq a, S_i \leq s|N, z) = C_0(a, s)$ for any $(a, s) \in [0, 1]^2$, and $U_0(\cdot)$ does not depend on N or z.
2. The entry cost function $c(z)$ is strictly increasing in z when $\mathcal{Z}$ is not a singleton, and is further continuous in z when $\mathcal{Z}$ is a closed interval with non-empty interior.

Exogenous variation in competition, either actual or potential, has been considered as a source of variation for testing and identification by many prior studies, including Haile, Hong, and Shum (2003) and GPV, among others. Exogenous variation in an entry instrument z follows GL, among others. While we analyze identification allowing for variation in both N and z, we expect that external instruments z may be challenging to find in practice.[5] For this reason, we will place particular emphasis on cases where only variation in N is available (or, equivalently, where $\mathcal{Z}$ is a singleton). Importantly, however, our identification results extend immediately to settings with asymmetric bidders, in which case types of i's rivals are also natural candidates for instruments affecting bidder i's entry but excludable (in the sense of Assumption 1) with respect to i's primitives.

In addition to the key exclusion restrictions in Assumption 1, we assume that (U_0, F_0, C_0) belong to regularity classes defined as follows:

Assumption 2. $U_0 \in \mathcal{U}$, where $\mathcal{U}$ is the set of utility functions $U(\cdot)$ such that:

1. $U : [0, \infty) \mapsto [0, \infty)$, $U(0) = 0$, and $U(1) = 1$.

ers (potential entrants) on each contract prior to the letting date. But only a small fraction of planholders actually submit bids (Li and Zheng (2009)), and the set of bids received is only disclosed after the letting concludes. We view such auctions as naturally modeled by the assumption of known N but unknown n. Empirical support for the assumption of unknown n is provided by Kong (2020), who shows in the context of New Mexico oil and gas auctions that even when $n = 1$ the single bidder typically bids well above the reserve. This finding is difficult to rationalize when n is known, but follows immediately when n is unknown.

[5] In Assumption 1, we make implicit use of the fact that z shifts *opportunity*, rather than financial, costs of entry: if instead z shifted financial entry costs, then z would affect the normalization of $U(\cdot)$ and one could not assume that $U_0(x)$ is invariant to z. This interpretation is consistent with the structural AS-RA application of Kong (2017), in which z measures oil and gas auctions outside the specific region considered. As pointed out by a referee, however, one may also be concerned that opportunities to bid in other auctions could affect bidder wealth, in which case z would best be treated as a covariate rather than an instrument.

2. $U(\cdot)$ is continuous on $[0,\infty)$ and admits three continuous derivatives on $(0,\infty)$, with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $(0,\infty)$.

3. Both $\lim_{x\downarrow 0} \frac{d}{dx}\left(\frac{U(x)}{U'(x)}\right)$ and $\lim_{x\downarrow 0} \frac{d^2}{dx^2}\left(\frac{U(x)}{U'(x)}\right)$ are finite.

Assumption 3. $F_0 \in \mathcal{F}$, where $\mathcal{F}$ is the set of probability distributions $F(\cdot)$ such that:

1. $F(\cdot)$ is supported on a compact interval $[\underline{v}, \bar{v}]$, with $\underline{v} \geq 0$ and $\bar{v} \in (\underline{v}, \infty)$.

2. $F(\cdot)$ is twice continuously differentiable with positive density on $[\underline{v}, \bar{v}]$.

Assumption 4. $C_0 \in \mathcal{C}$, where $\mathcal{C}$ is the set of bivariate copula functions $C(a,s)$ such that, interpreted as a distribution over random variables (A,S) with uniform marginals:

1. $C(a,s)$ is continuous on $[0,1]\times[0,1]$.

2. For all $s \in [0,1)$, the distribution of A given $S \geq s$ admits a continuous, bounded density with infimum support $\underline{a}(s)$ continuous in s, and for all points in its support except possibly the infimum $\underline{a}(s)$, this density is locally bounded away from zero, differentiable in a, and differentiable in s.

3. For all $a \in [0,1]$, $C(a,s)$ is concave in s.

Assumptions 2 and 3 impose standard regularity conditions on U_0 and F_0, following GPV among others. Assumptions 4.1-4.2 ensure that regularity conditions on F_0 pass through to the distributions of V_i conditional on $S_i \geq s$ which arise in equilibrium, while nesting Samuelson (1985)'s model of perfectly selective entry within the class $\mathcal{C}$.[6] Finally, Assumption 4.3 implies that V_i is weakly increasing in S_i in the sense of first-order stochastic dominance; this can be seen most readily when $C_0(a,s)$ is differentiable, in which case $F_0(v|S_i = s) \equiv \frac{\partial C_0(F_0(v),s)}{\partial s}$. We maintain Assumptions 1-4 throughout the analysis.

As in GPV, rather than working with U_0, it will frequently prove more convenient to consider the one-to-one transformation $\lambda_0(x) \equiv \frac{U_0(x)}{U_0'(x)}$. In view of the normalizations above, $U_0(x) = \exp(\int_1^x 1/\lambda_0(t)\,dt)$. Further, since $U_0(0) = 0$, $U_0' > 0$, and $U_0'' \leq 0$, we have both $\lambda_0(0) = 0$ and $\lambda_0'(x) = 1 - \frac{U_0(x)}{U_0'(x)}\frac{U_0''(x)}{U_0'(x)} \geq 1$. It follows that $\lambda_0(\cdot)$ has a well-defined, monotone inverse $\lambda_0^{-1}(\cdot)$ satisfying $\lambda_0^{-1}(0) = 0$ and $\lambda_0^{-1,\prime} \leq 1$. We will work with U_0, λ_0, and λ_0^{-1} interchangeably. Let Λ be the set of functions $\lambda(x)$ such that $\lambda(x) \equiv [U(x)/U'(x)]$ for some $U \in \mathcal{U}$, and Λ^{-1} be the set of functions λ^{-1} which are inverses of some function $\lambda \in \Lambda$.

[6]Formally, this model is nested by setting $C(a,s) = \min(a,s)$, in which case $C(a,s)$ does not admit a joint density but does satisfy the smoothness conditions in Assumption 4, which are sufficient for our results.

2.3 Equilibrium behavior

We focus on the unique symmetric, monotone equilibrium of the AS-RA model with general boundary conditions. See Appendix A for a complete derivation of this equilibrium, in a more general setting additionally accommodating nonzero initial wealth and financial costs of entry. We describe only its identification-relevant features here.

For each competition level $k \in \mathcal{K}$ and each instrument level $z \in \mathcal{Z}$, there will be a unique signal threshold $s_k(z) \in [0,1]$ such that potential bidder j enters when $S_j \geq s_k(z)$. If $s_k(z) = 0$, then all potential bidders enter with certainty. Meanwhile, if $s_k(z) = 1$, then no bidder enters and no bids are observed. We are primarily interested in interior entry ($s_k(z) \in (0,1)$), although we also consider full entry ($s_k(z) = 0$) for completeness.

Given a symmetric entry threshold $s_k(z) \in [0,1)$, the distribution of valuations among bidders choosing to enter conditional on observables (N_k, z) will be described by the c.d.f.

$$F_k^0(v|z) \equiv F_0(v|S_j \geq s_k(z)) = \frac{F_0(v) - C_0(F_0(v), s_k(z))}{1 - s_k(z)}, \quad k = 1, ..., K. \tag{1}$$

Let $v_0(\cdot) : [0,1] \mapsto [\underline{v}, \bar{v}]$ be the quantile function of the ex ante value distribution function $F_0(\cdot) : [\underline{v}, \bar{v}] \mapsto [0,1]$. For any $z \in \mathcal{Z}$, we let $v_k(\cdot|z)$ be the quantile function of the post-entry value distribution $F_k^0(\cdot|z)$. When $\mathcal{Z}$ is a singleton and the conditioning on z becomes trivial, we will simply use $F_k^0(v)$ and $v_k(\alpha)$ (for $\alpha \in [0,1]$) to denote $F_k^0(v|z)$ and $v_k(\alpha|z)$ respectively.

Taking the entry threshold $s_k(z) \in [0,1)$ as given, bidding at competition level $k \in \mathcal{K}$ will be described by a symmetric, monotone strategy $\beta_k(\cdot|z)$ such that entrant i drawing value v_i optimally submits bid $\beta_k(v_i|z)$. In equilibrium, entrant i submitting bid $\beta_k(y|z)$ expects to outbid any potential rival j in one of two events: either j does not enter (probability $s_k(z)$), or j does enter but draws a valuation below y (probability $(1 - s_k(z))F_k^0(y|z)$). When potential rivals play equilibrium strategies, we may therefore write i's bidding problem as

$$\max_y \left\{ U_0(v_i - \beta_k(y|z)) \cdot [s_k(z) + (1 - s_k(z))F_k^0(y|z)]^{N_k - 1} \right\}.$$

Taking a first-order condition with respect to y and enforcing the equilibrium condition $y = v_i$, we obtain the following differential equation characterizing $\beta_k(\cdot|z)$:

$$\beta_k'(v|z) = \lambda_0(v - \beta_k(v|z)) \frac{(N_k - 1)(1 - s_k(z)) f_k^0(v|z)}{s_k(z) + (1 - s_k(z))F_k^0(v|z)}, \tag{2}$$

where $\beta_k'(v|z)$ and $f_k^0(v|z)$ denote, respectively, the derivatives of $\beta_k(v|z)$ and $F_k^0(v|z)$ with respect to v. It only remains to specify appropriate boundary conditions for $\beta_k(v|z)$. These will be determined by the bidder with the lowest type $v_i = v_k(0|z)$, whose behavior will depend crucially on whether $s_k(z) = 0$ or $s_k(z) \in (0,1)$.

If $s_k(z) = 0$, then the bidder with the lowest type wins with probability zero and thus cannot do better than to bid their value. Thus, noting that $s_k(z) = 0$ also implies $v_k(0|z) = v_0(0) = \underline{v}$, the relevant boundary condition when $s_k(z) = 0$ is $\beta_k(\underline{v}|z) = \underline{v}$. Meanwhile, if $s_k(z) \in (0,1)$, then an entrant with type $v_i = v_k(0|z)$ will still win if they are the only bidder, which occurs with probability $s_k(z)^{N_k-1} > 0$. Conditional on being the only bidder, it is obviously optimal to bid the reserve price p_0. Thus, when $s_k(z) \in (0,1)$, the relevant boundary condition is $\beta_k(v_k(0|z)|z) = p_0$. In either case, Equation (2) together with the relevant boundary condition will uniquely determine $\beta_k(\cdot|z)$.

Finally, consider the threshold $s_k(z)$ characterizing equilibrium entry at (N_k, z). Note that if $p_0 < \underline{v}$, the minimum equilibrium bid will in fact change discontinuously as $s_k(z) \to 0^+$. We show in Theorem 4 in Appendix A, however, that for every fixed type $v_i > \underline{v}$, the bid submitted by type v_i as $s_k(z) \to 0^+$ will converge to that submitted by type v_i at $s_k(z) = 0$. We further show that bid functions are increasing and continuous in $s_k(z)$ for all $s_k(z) \in (0,1)$. These properties ensure that pre-entry profits are increasing and continuous in $s_k(z)$, which in turn implies existence of a unique equilibrium entry threshold $s_k(z) \in [0,1]$ which is increasing and continuous in $c(z)$. In particular, if $s_k(z) \in (0,1)$, then a potential bidder with signal $S_i = s_k(z)$ must be indifferent between entering and remaining out when facing $N_k - 1$ potential rivals who play equilibrium strategies. Furthermore, for all $k \in \mathcal{K}$, if $s_k(z) \in (0,1)$, then $s_k(z)$ is strictly increasing in both N_k and z, and is continuous in z whenever z is a continuous instrument. Finally, if $s_1(z) < 1$, then $s_k(z) < 1$ for all $k \in \mathcal{K}$.

2.4 Linking observables to unobservables

For each $k \in \mathcal{K}$ and $z \in \mathcal{Z}$, let $G_k(b|z)$ be the equilibrium distribution of bids submitted at (N_k, z), $g_k(b|z)$ be the density associated with $G_k(b|z)$, and $b_k(\alpha|z)$ be the quantile function associated with $G_k(b|z)$. As usual, observing bids will (point) identify $G_k(\cdot|z)$ for each $k \in \mathcal{K}$ and $z \in \mathcal{Z}$. Similarly, recalling $S_i \sim U[0,1]$, we may (point) identify the equilibrium entry threshold $s_k(z)$ for each (k, z) from observed probabilities of entry:

$$s_k(z) = 1 - \frac{E[n|N_k, z]}{N_k}.$$

We next derive the key equilibrium inverse bidding function linking the directly identified objects $s_1(z), ..., s_K(z)$, $G_1(\cdot|z), ..., G_K(\cdot|z)$ to latent bid-stage primitives. Toward this end, consider any $z \in \mathcal{Z}$ such that $s_1(z) < 1$ (and recall that this implies $s_k(z) < 1$ for all $k \in \mathcal{K}$). Following GPV, we first apply the change of variables $b_i = \beta_k(v_i|z)$ to the first-order condition (2), then exploit strict monotonicity of b_i in v_i to re-express both bids and values in terms of their respective quantile functions $b_k(\alpha|z)$ and $v_k(\alpha|z)$. These transformations ultimately yield the following equilibrium quantile inverse bidding functions:

$$v_k(\alpha|z) = b_k(\alpha|z) + \lambda_0^{-1}(R_k(\alpha|z)), \quad \text{for all } \alpha \in [0,1],\ k \in \mathcal{K},\ z \in \mathcal{Z}, \tag{3}$$

where the argument $R_k(\alpha|z)$ to the unknown function $\lambda_0^{-1}(\cdot)$ is defined as

$$R_k(\alpha|z) \equiv \left[\frac{s_k(z) + (1 - s_k(z))\alpha}{(N_k - 1)(1 - s_k(z))}\right] b_k'(\alpha|z). \tag{4}$$

For each $z \in \mathcal{Z}$ such that $s_k(z) \in (0,1)$, properties of $\beta_k(\cdot|z)$ imply that $b_k(0|z) = p_0$, that $b_k(\cdot|z)$ is differentiable on its domain with $b_k'(\alpha|z) = [g_k(b_k(\alpha|z)|z)]^{-1}$, and that $R_k(\cdot|z)$ is continuous on $[0,1]$ and differentiable on at least $(0,1]$.[7] Definition (4) also shows that when either $s_k(z) > 0$ or $\alpha > 0$, $R_k(\alpha|z)$ is continuous in $z \in \mathcal{Z}$ when z is a continuous instrument. Furthermore, (point) identification of $s_k(z), G_k(\cdot|z)$ implies (point) identification of $b_k(\cdot|z)$ and $R_k(\cdot|z)$, and hence (point) identification of the right-hand side of (3) up to the unknown $\lambda_0^{-1}(\cdot)$. Finally, note that (3) can substantively restrict $\lambda_0^{-1}(\cdot)$ on, at most, its equilibrium domain $[\underline{r}, \bar{r}]$, where $\underline{r} \equiv \inf_{k\in\mathcal{K}, z\in\mathcal{Z}} \min_{\alpha\in[0,1]} R_k(\alpha|z)$ and $\bar{r} \equiv \sup_{k\in\mathcal{K}, z\in\mathcal{Z}} \max_{\alpha\in[0,1]} R_k(\alpha|z)$.

3 Identification of bid-stage primitives

This section first provides a sharp nonparametric characterization of restrictions on bid-stage primitives $(\lambda_0^{-1}, F_0, C_0)$ generated by the bid distributions $G_1(\cdot|z), ..., G_K(\cdot|z)$, taking entry thresholds $s_1(z)$, ..., $s_K(z)$ as given (Theorem 1).[8] We then present several important implications of this sharp characterization. We show that risk aversion is nonparametrically testable in the sense that, given variation in either N or z, risk neutrality is outside the identified set when bidders are strictly risk averse. We also establish results on point identification based on variation in both N and a continuous z, identification with a parametric utility, and conditional identification with a parametric copula.

3.1 Nonparametric bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$

We begin by analyzing the nonparametric bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$, denoted by $\mathcal{I}$ and defined formally as follows:

[7] If $\lim_{a\to 0} v_k'(a|z) < \infty$, then $R_k(\cdot|z)$ is differentiable on $[0,1]$.

[8] Our focus on bid-stage primitives is motivated by two findings of GL in their risk neutral (i.e., $\lambda_0^{-1}(x) = x$) context. First, GL show how to map restrictions on (F_0, C_0) implied by bidding into identified sets for $c(z)$; Second, they find that entry-stage restrictions convey little additional information on bid-stage primitives. In Appendix A.3 of our previous version SSRN-3681530 (May 28, 2023), we translate our sharp identified set for bid-stage primitives $(\lambda_0^{-1}, F_0, C_0)$ into bounds on conditional distributions and entry costs. But, in view of GL's second finding and due to the lack of space, we focus on restrictions on bid-stage primitives implied by equilibrium bidding, taking observed entry patterns as given.

Definition 1. The bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$, denoted by $\mathcal{I} \subset \Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$, is the set of all $(\lambda^{-1}, F, C) \in \Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$ that jointly satisfy equations (1) and (3) for all $k \in \mathcal{K}$ and all $z \in \mathcal{Z}$.

Equivalently, $\mathcal{I}$ is the subset of $\Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$ such that, for all $k \in \mathcal{K}$ and $z \in \mathcal{Z}$, $G_k(\cdot|z)$ is the equilibrium bid distribution implied by each $(\lambda^{-1}, F, C) \in \mathcal{I}$ given $s_k(z)$.

We next provide a sharp characterization of $\mathcal{I}$, emphasizing restrictions on λ^{-1} generated by equilibrium bidding. Toward this end, consider any candidate $\lambda^{-1} \in \Lambda^{-1}$. Under the hypothesis $\lambda^{-1} = \lambda_0^{-1}$, the quantile inverse bidding function (3) implies a unique set of candidates $\tilde{v}_1(\cdot|z;\lambda^{-1}), ..., \tilde{v}_K(\cdot|z;\lambda^{-1})$ for the unknown latent quantile functions $v_1(\cdot|z), ..., v_K(\cdot|z)$:

$$\tilde{v}_k(\alpha|z;\lambda^{-1}) \equiv b_k(\alpha|z) + \lambda^{-1}(R_k(\alpha|z)), \quad \text{for all } \alpha \in [0,1],\ k \in \mathcal{K},\ z \in \mathcal{Z}\ . \tag{5}$$

By construction, these candidates $\tilde{v}_1(\cdot|z;\lambda^{-1}), ..., \tilde{v}_K(\cdot|z;\lambda^{-1})$ are identified up to λ^{-1} and well-defined for any $\lambda^{-1} \in \Lambda^{-1}$. Furthermore, properties of $b_k(\cdot|z)$ and $R_k(\cdot|z)$ imply that, for all $\lambda^{-1} \in \Lambda^{-1}$, $\tilde{v}_k(\cdot|z;\lambda^{-1})$ is differentiable on the same domain as $v_k(\cdot|z)$.

Next observe that taking $\lambda_0^{-1} \in \Lambda^{-1}$ and entry behavior as given, primitives $(F_0, C_0) \in \mathcal{F} \times \mathcal{C}$ influence bidding behavior only through the latent quantile functions $v_1(\cdot|z), ..., v_K(\cdot|z)$ (see (1)). To determine whether any candidate $\lambda^{-1} \in \Lambda^{-1}$ is consistent with bid-stage observables, it is therefore sufficient to determine whether there exists a structure $(F, C) \in \mathcal{F} \times \mathcal{C}$ consistent with the candidate quantile functions $\tilde{v}_1(\cdot|z;\lambda^{-1}), ..., \tilde{v}_K(\cdot|z;\lambda^{-1})$ generated by λ^{-1} through (5). This in turn reduces to a set of five restrictions on $\tilde{v}_1(\cdot|z;\lambda^{-1}), ..., \tilde{v}_K(\cdot|z;\lambda^{-1})$, yielding the following sharp characterization of the bid-stage identified set $\mathcal{I}$:

Theorem 1. *Let Λ_I^{-1} be the set of $\lambda^{-1} \in \Lambda^{-1}$ such that the candidate quantile functions $\tilde{v}_k(\cdot|z;\lambda^{-1})$ defined by (5) satisfy all of the following restrictions **M**, **O**, **I**, **D** and **S**:*

M *For all $k \in \mathcal{K}$, $z \in \mathcal{Z}$, and all $a \in (0,1]$, $\tilde{v}'_k(a|z;\lambda^{-1})$ is bounded away from zero.*

O *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $s_k(z) \leq s_l(z')$, $\tilde{v}_k(a|z;\lambda^{-1}) \leq \tilde{v}_l(a|z';\lambda^{-1})$ for all $a \in [0,1]$, with equality if $s_k(z) = s_l(z')$.*

I *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$, $\tilde{v}_k(1|z;\lambda^{-1}) = \tilde{v}_l(1|z';\lambda^{-1})$.*

D *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $s_k(z) \leq s_l(z')$, and all $y, y' \in \mathbb{R}$ with $y' \geq y$,*

$$\big(1-s_k(z)\big)\left[\tilde{v}_k^{-1}(y'|z;\lambda^{-1}) - \tilde{v}_k^{-1}(y|z;\lambda^{-1})\right] \geq \big(1-s_l(z')\big)\left[\tilde{v}_l^{-1}(y'|z';\lambda^{-1}) - \tilde{v}_l^{-1}(y|z';\lambda^{-1})\right].$$

S *For all $k, l, m \in \mathcal{K}$ and $z, z', z'' \in \mathcal{Z}$ such that $s_k(z) < s_l(z') < s_m(z'')$, and all $y \in \mathbb{R}$,*

$$\frac{(1-s_k(z))\,\tilde{v}_k^{-1}(y|z;\lambda^{-1}) - (1-s_l(z'))\,\tilde{v}_l^{-1}(y|z';\lambda^{-1})}{s_l(z') - s_k(z)} \geq \frac{(1-s_l(z'))\tilde{v}_l^{-1}(y|z';\lambda^{-1}) - (1-s_m(z''))\tilde{v}_m^{-1}(y|z'';\lambda^{-1})}{s_m(z'') - s_l(z')}.$$

Then for any $\lambda^{-1} \in \Lambda^{-1}$, the following two statements are equivalent: (i) $\lambda^{-1} \in \Lambda_I^{-1}$, (ii) there exists $(F, C) \in \mathcal{F} \times \mathcal{C}$ such that $(\lambda^{-1}, F, C) \in \mathcal{I}$. Moreover, if $(\lambda^{-1}, F, C) \in \mathcal{I}$, then for all $k \in \mathcal{K}$ and $z \in \mathcal{Z}$,

$$\tilde{v}_k^{-1}(y|z;\lambda^{-1}) = \frac{F(y) - C(F(y), s_k(z))}{1 - s_k(z)} \text{ for all } y \in [\tilde{v}_k(0|z;\lambda^{-1}), \tilde{v}_k(1|z;\lambda^{-1})].$$

Note that Theorem 1 allows $\Lambda_I^{-1} = \emptyset$, in which case there exists no symmetric AS-RA model rationalizing bid-stage behavior.[9] Restrictions M-S of Theorem 1 reflect properties of $F_1(\cdot|z), ..., F_K(\cdot|z)$ implied by $(F, C) \in \mathcal{F} \times \mathcal{C}$. Restriction M (strict monotonicity) follows since each density $f_k(\cdot|z)$ is bounded. Restriction O (ordered quantile functions) reflects the fact that entrant values are stochastically increasing in $s_k(z)$. Restriction I (invariant top quantile) follows from stochastic ordering of V_i in S_i, together with the fact that so long as $s_k(z) < 1$, the set of entering types will include the potential bidder drawing the highest possible signal ($S_i = 1$). Restriction D (positive conditional densities) can be understood by noting that the c.d.f. of V_i given $S_i \in [s_k(z), s_l(z')]$ is proportional to $(1 - s_k(z))F_k(\cdot|z) - (1 - s_l(z'))F_l(\cdot|z')$. Restriction S (stochastically increasing conditional distributions) follows since, if V_i is stochastically increasing in S_i, then any conditional c.d.f. of the form $F(V_i|S_i \in [s, s'])$ must be decreasing in both s and s'. The final statement implies that the selected distributions $F_1(\cdot|z), ..., F_K(\cdot|z)$ are identified up to λ^{-1}, although as in GL this typically does not imply identification of (F, C) up to λ^{-1}.[10]

The following subsections present several important implications of Theorem 1.

3.2 Nonparametric testability of risk neutrality

In this section, we show that risk neutrality is nonparametrically testable within the AS-RA model based on variation in either N or z (Corollary 1). Moreover, with continuous variation in z, restrictions on the form of risk preferences also become testable (Corollary 2). Both corollaries follow from Restriction I (invariant top quantile) of Theorem 1, which substituted into equation (5) implies for all $\lambda^{-1} \in \Lambda_I^{-1}$:

$$b_l(1|z') - b_k(1|z) = \lambda^{-1}(R_k(1|z)) - \lambda^{-1}(R_l(1|z')) \quad \forall k, l \in \mathcal{K},\ z, z' \in \mathcal{Z}. \tag{6}$$

Recall that weak risk aversion ($U_0'' \leq 0$) implies $\lambda_0' \geq 1$ and hence $0 \leq \lambda_0^{-1,\prime} \leq 1$. Global risk neutrality ($U_0'' = 0$) corresponds to the special case $\lambda_0^{-1,\prime}(x) = 1$ for all x. We say that bidders are *strictly risk averse* at x if $U_0''(x) < 0$, or equivalently if $\lambda_0^{-1,\prime}(x) < 1$.

[9] As in GPV (2000, 2009) and elsewhere, the simple IPV framework we take as our expositional focus also implies symmetry and independence of entry and bidding choices across bidders conditional on N_k, z.

[10] Our sharp nonparametric identified set characterization contributes to a rapidly growing literature on partial identification in structural models; see, for example, Manski and Tamer (2002), Haile and Tamer (2003), Molinari (2020) and the references therein.

First consider any $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $R_l(1|z') < R_k(1|z)$. If bidders are strictly risk averse for some $x \in [R_l(1|z'), R_k(1|z)]$, i.e. $\lambda_0^{-1,\prime}(x) < 1$, then (6) implies that $b_l(1|z') - b_k(1|z) < R_k(1|z) - R_l(1|z')$. Conversely, if bidders are globally risk neutral, then *any* candidate $\lambda^{-1} \in \Lambda_I^{-1}$ must satisfy $\int_{R_l(1|z')}^{R_k(1|z)} \lambda^{-1,\prime}(r)\, dr = R_k(1|z) - R_l(1|z')$. Recalling that $\lambda_0^{-1,\prime}(x) \leq 1$, these facts imply that risk neutrality is testable in the following strong sense:

Corollary 1. *Let* $\bar{R}(1) \equiv \sup_{k \in \mathcal{K}, z \in \mathcal{Z}} R_k(1|z)$ *and* $\underline{R}(1) \equiv \inf_{k \in \mathcal{K}, z \in \mathcal{Z}} R_k(1|z)$. *Then:*

1. *If bidders are risk neutral, then any* $\lambda^{-1} \in \Lambda_I^{-1}$ *must satisfy* $\lambda^{-1,\prime}(x) = 1$ *for all* $x \in [\underline{R}(1), \bar{R}(1)]$*;*
2. *If, for any* $x \in [\underline{R}(1), \bar{R}(1)]$*, bidders are strictly risk averse at* x*, then no risk neutral model can rationalize bid-stage behavior.*

Now further suppose that $z \in \mathcal{Z}$ is a *continuous* instrument. In this case, provided that $s_k(z) > 0$, variation in z will induce continuous variation in $R_k(1|z)$. Extending the arguments above, this allows us to point identify $\lambda_0^{-1,\prime}(r)$ for at least some r:

Corollary 2. *Suppose that* $\mathcal{Z} = [\underline{z}, \bar{z}]$ *with* $\bar{z} > \underline{z}$*, and consider any* $k \in \mathcal{K}$ *such that* $s_k(z) \in (0,1)$ *for some* $z \in \mathcal{Z}$*. Then* $R_k(1|\bar{z}) > R_k(1|\underline{z})$*, and* $\lambda_0^{-1,\prime}(r)$ *is identified for all* $r \in [R_k(1|\underline{z}), R_k(1|\bar{z})]$*.*

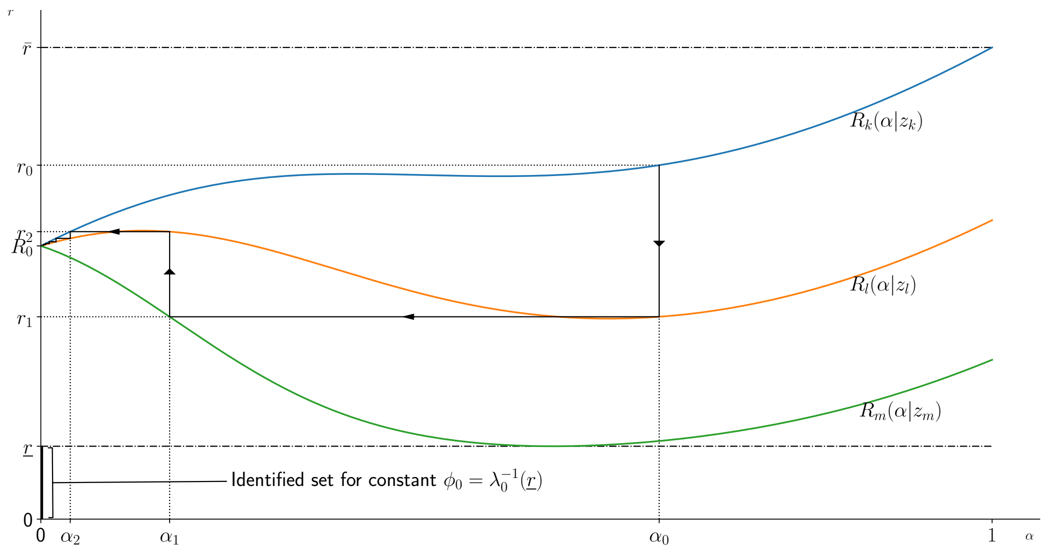
Note that, differentiating the definition $\lambda_0(x) \equiv U_0(x)/U_0'(x)$, we can express the Arrow-Pratt coefficient of absolute risk aversion as $ARA(\lambda_0^{-1}(r)) = \frac{1}{r\lambda_0^{-1,\prime}(r)} - \frac{1}{r}$. Thus, with a continuous instrument z, restrictions on the form of risk aversion further become testable.

Corollaries 1 and 2 turn only on an invariant top quantile of values among entrants (Restriction I of Theorem 1). Restriction I is here a consequence of the assumption that valuations are stochastically increasing in signals, together with the fact that the set of entering types (if nonempty) will always include the potential bidder with the highest possible signal ($S_i = 1$). Importantly, however, a similar insight applies in any first-price auction where at least one quantile of values is invariant to either N or z. CGPV have considered parametric quantile restrictions, including quantile invariance, as a basis for estimation with parametric U_0. To our knowledge, however, the fact that quantile invariance also implies nonparametric testability of risk neutrality has not been previously observed.

3.3 Identification with both variation in N and a continuous z

Finally, suppose that we have both variation in N and a continuous instrument z. In this case, we may be able to find $N_k < N_l < N_m$ and $z_k, z_l, z_m \in \mathcal{Z}$ such that $s_k(z_k) = s_l(z_l) = s_m(z_m)$; i.e., three competition levels such that potential bidders enter with the same probabilities despite facing different numbers of rivals. As we show next, λ_0^{-1} is then identified on its

Figure 1: Constructing a recursive sequence to identify $\lambda_0^{-1}(r)$ based on three competition structures (N_k, z_k), (N_l, z_l), (N_m, z_m) such that $s_k(z_k) = s_l(z_l) = s_m(z_m)$.



relevant equilibrium domain, up to a constant representing the value of λ_0^{-1} at its minimum equilibrium argument. In contrast to GPV, who show that without entry two competition levels are both necessary and sufficient for identification, we also show that unless the reserve price p_0 is at least weakly binding, only two competition levels may not suffice.

Toward this end, consider any (N_k, z_k), (N_l, z_l), (N_m, z_m) with $N_k < N_l < N_m$ but $s_k(z_k) = s_l(z_l) = s_m(z_m) \in (0, 1)$. By Restriction O of Theorem 1, since $s_k(z_k) = s_l(z_l) = s_m(z_m)$, we must also have $v_k(\cdot|z_k) = v_l(\cdot|z_l) = v_m(\cdot|z_m)$. Consequently, similar to GPV, we can substitute from (3) to obtain the following system of equations: for all $a \in [0, 1]$,

$$b_k(a|z_k) + \lambda_0^{-1}(R_k(a|z_k)) = b_l(a|z_l) + \lambda_0^{-1}(R_l(a|z_l)) = b_m(a|z_m) + \lambda_0^{-1}(R_m(a|z_m)). \qquad (7)$$

Since $v_k(\cdot|z_k) = v_l(\cdot|z_l) = v_m(\cdot|z_m)$, the functions $R_k(\cdot|z_k)$, $R_l(\cdot|z_l)$, $R_m(\cdot|z_m)$ must satisfy the following properties. First, since bidders with $V_i > v_k(0|z_k)$ will bid more aggressively against more expected competition, $R_k(a|z_k) > R_l(a|z_l) > R_m(a|z_l) > 0$ for all $a \in (0, 1]$. Second, since when $s_k(z_k) > 0$ a bidder with $v_i = v_k(0|z_k)$ will bid p_0, $R_k(0|z_k) = R_l(0|z_l) = R_m(0|z_m) = \lambda_0(v_l(0|z_l) - p_0)$. In particular, letting $R_0 \equiv R_k(0|z_k) = R_l(0|z_l) = R_m(0|z_m)$ and assuming $s_k(z_k) \in (0, 1)$, we will have $R_0 = 0$ if $p_0 = v_k(0|z_k)$, but $R_0 > 0$ if $p_0 < v_k(0|z_k)$.

Now let $\underline{R}_m(z_m) = \min_a R_m(a|z_m)$ and $\bar{R}_k(z_k) = \max_a R_k(a|z_k)$; these are the smallest and largest arguments to λ_0^{-1} observed across (N_k, z_k), (N_l, z_l), (N_m, z_m). Consider any $r_0 \in [\underline{R}_m(z_m), \bar{R}_k(z_k)]$. If $R_l(\alpha_0|z_l) \geq R_0$, then since $R_k(\alpha_0|z_k) > R_l(\alpha_0|z_l)$ and $R_0 \equiv R_k(0|z_k)$ we can find $\alpha_1 \in [0, \alpha_0)$ such that $R_k(\alpha_1|z_k) = R_l(\alpha_0|z_l)$. Meanwhile, if $R_l(\alpha_0|z_l) < R_0$, then since $R_l(\alpha_0|z_l) > R_m(\alpha_0|z_m)$ and $R_0 \equiv R_m(0|z_m)$ we can find $\alpha_1 \in (0, \alpha_0)$ such that

$R_m(\alpha_1|z_m) = R_l(\alpha_0|z_l)$. As illustrated in Figure 1, we can apply these cases recursively to construct an identified sequence $\{\alpha_j\}_{j=0}^{\infty}$ such that $\alpha_j \to 0$ and at every step j, we have either $R_k(\alpha_{j+1}|z_k) = R_l(\alpha_j|z_l)$ or $R_m(\alpha_{j+1}|z_m) = R_l(\alpha_j|z_l)$. Letting $\iota_j \in \{k, m\}$ denote the competition level used to define α_j at each step j, we can then substitute recursively into (7) to express $\lambda_0^{-1}(r_0)$ as an identified sum of bid differences plus a trailing constant:

$$\lambda_0^{-1}(r_0) = \sum_{j=0}^{\infty} \left[b_l(\alpha_j|z_l) - b_{\iota_j}(\alpha_j|z_{\iota_j}) \right] + \lambda_0^{-1}(R_0).$$

When $p_0 < v_k(0|z_k)$ and thus $R_0 > 0$, the trailing constant will not vanish (in contrast to GPV). Re-normalizing terms, however, we can express this constant as identified up to the value of $\lambda_0^{-1}(r)$ at the minimum argument $r = \underline{R}_m(z_m)$ observed across the competition structures $(N_k, z_k), (N_l, z_l), (N_m, z_m)$. We thereby obtain the following result:

Theorem 2. *Let $k, l, m \in \mathcal{K}$ be any three distinct competition levels, $N_k < N_l < N_m$, and suppose that there exist $z_k, z_l, z_m \in \mathcal{Z}$ such that $s_k(z_k) = s_l(z_l) = s_m(z_m)$. Then for all $r \in [\underline{R}_m(z_m), \bar{R}_k(z_k)]$, $\lambda_0^{-1}(r) = \phi_1(r) + \phi_2$, where $\phi_1(r)$ is an identified function such that $\phi_1(\underline{R}_m(z_m)) = 0$, and $\phi_2 \in [0, \underline{R}_m(z_m)]$ is an interval-identified constant.*

Obviously, equilibrium bidding cannot restrict $\lambda_0^{-1}(\cdot)$ outside its equilibrium domain, so this result is the strongest we can expect. We have also verified that for any $R_0 > 0$, there exist cases, such as that illustrated in Figure 1, where two competition levels in fact do not suffice.[11] On the other hand, if $R_0 = 0$, then we trivially have both $\underline{R}_m(z_m) = 0$ and $R_l(\alpha_0|z_l) > R_0$ for all $\alpha_0 > 0$. Consequently, $\phi_2 = 0$, only the first case above is ever employed, and two competition levels suffice for point identification as in GPV.

Importantly, even with a continuous instrument z, point identification of ϕ_2 follows only if there exists a sequence of observables such that $\underline{R}_m(z_m) \to 0$. Even with $R_0 > 0$, this is possible in principle; for example, if for some $s_k(z_k) > 0$ there exists a sequence of (N_m, z_m) with $N_m \to \infty$ and $s_m(z_m) = s_k(z_k)$ for all m, then $\underline{R}_m(z_m) \to 0$. In practice, however, we expect that ϕ_2 will typically be set identified unless $R_0 = 0$ for some auctions in the population. In the symmetric AS-RA model with incomplete entry ($s_k(z_k) > 0$) which is our focus here, this in turn holds if and only if $p_0 = v_k(0|z_k)$ for some auctions in the population. Such just-binding reserve prices could arise either through strategic choice by the auctioneer, or through a stochastic public reserve price whose support includes $v_k(0|z_k)$. $R_0 = 0$ would also hold in models where bidders are asymmetric and some bidder enters with certainty (e.g., mills in Athey, Levin, and Seira (2011)), or where the auctioneer acts as a bidder of last resort in the event of insufficient entry (e.g. Li and Zheng (2009)). Although we do not model these extensions formally, our identification arguments apply to them directly.

[11] Indeed, if $v_k'(0|z_k) < \infty$, then simply by additively shifting valuations we can always find examples where for all α_0 sufficiently close to zero, there is no $\alpha_1 < \alpha_0$ such that $R_k(\alpha_1|z_k) = R_l(\alpha_0|z_l)$.

3.4 Point identification of λ_0^{-1} with a parametric utility

In some applications, one may be willing to assume that U_0 belongs to a parametric family: i.e., that $\lambda_0^{-1} = \lambda^{-1}(\cdot;\gamma_0)$ for some $\gamma_0 \in \Gamma$, with Γ being a compact subset of a finite dimensional Euclidean space. Theorem 1 may then imply point identification of λ_0^{-1}, although potentially only set identification of other bid-stage primitives.

To see this, recall from Restriction I (invariant top quantile) of Theorem 1 that $\gamma = \gamma_0$ implies that $\tilde{v}_k(1|z;\gamma)$ is constant for all k and z. Taking $\bar{v} \equiv \tilde{v}_k(1|z;\gamma)$ as an auxiliary parameter to be identified, we may equivalently express this restriction as

$$\bar{v} = b_k(1|z) + \lambda^{-1}(R_k(1|z);\gamma_0), \quad \forall\, k \in \mathcal{K},\, z \in \mathcal{Z}. \tag{8}$$

This parallels the system of estimating restrictions considered by CGPV, here derived directly from AS entry. If the system (8) has a unique solution $(\gamma_0, \bar{v})$, then identification of $\lambda_0^{-1} = \lambda^{-1}(\cdot;\gamma_0)$ is immediate, with identification of $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ following through (5). Uniqueness of $(\gamma_0, \bar{v})$ will depend on both the parametric family $\{\lambda^{-1}(\cdot;\gamma) : \gamma \in \Gamma\}$ and the scope of variation in (N, z). If, however, U_0 belongs to either the Constant Relative Risk Aversion (CRRA) or the Constant Absolute Risk Aversion (CARA) families, identification based on (8) can be shown analytically:

Corollary 3. *Assume that U_0 belongs to either of the following parametric families:*

CRRA $U_0(x) = x^{1-\rho_0}$*; $\rho_0 \in [0, \bar{\rho}]$ for some $\bar{\rho} < 1$.*

CARA $U_0(x) = x$ *for $\gamma_0 = 0$; $U_0(x) \propto (1 - e^{-\gamma_0 x})$ for $\gamma_0 \in (0, \bar{\gamma}]$ with $\bar{\gamma} < \infty$.*

Further suppose that $R_k(1|z) \neq R_l(1|z')$ for some $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$. Then λ_0^{-1} and $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ are point identified.

Depending on the scope of variation in z, point identification of $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ may or may not be sufficient to point-identify F_0 and C_0. Given point identification of λ_0^{-1} and $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$, however, we may construct identified sets for F_0 and C_0, as well as for the entry cost function $c(z)$, following GL.

3.5 Conditional identification with a parametric copula

In addition, or as an alternative, to parameterizing utility, one may assume that C_0 belongs to a known parametric family: i.e., $C_0(a, s) = C(a, s; \theta_0)$, with θ_0 an element of a compact subset Θ of some Euclidean space.[12] We further assume that if $(A, S) \sim C(a, s; \theta)$, the distribution

[12] Parametric copula assumptions have also been proposed to correct for selection in other contexts, see e.g. Arellano and Bonhomme (2017).

of A given $S \geq s$ has support $[0,1]$ for all $s \in [0,1)$; this ensures that $v_k(0|z) = \underline{v}$ for all $k \in \mathcal{K}$, $z \in \mathcal{Z}$. As we show next, (λ^{-1}, F) are then generically identified (up to a constant) conditional on θ. In other words, $\mathcal{I}$ can be indexed (up to a constant) by θ.

For simplicity, we focus on variation in N_k only, assuming $K \geq 3$. Let Θ_I denote the bid-stage identified set for θ_0: i.e., the set of $\theta \in \Theta$ for which there exist some $(\lambda^{-1}, F) \in \Lambda^{-1} \times \mathcal{F}$ such that $(\lambda^{-1}, F, C(\cdot,\cdot;\theta)) \in \mathcal{I}$. For each $k \in \mathcal{K}$, define $h_k : [0,1] \times \Theta \mapsto [0,1]$ as follows:

$$h_k(a;\theta) \equiv \frac{a - C(a, s_k; \theta)}{1 - s_k}. \tag{9}$$

Note that each function $h_k(\cdot;\theta)$ is identified up to the unknown copula parameter θ. Furthermore, at $\theta = \theta_0$, we have from (1) that for each $k \in \mathcal{K}$,

$$F_k^0(y) = \frac{F_0(y) - C(F_0(y), s_k; \theta_0)}{1 - s_k} \equiv h_k(F_0(y); \theta_0). \tag{10}$$

Applying the change of variables $y = v_0(a)$ on both sides of (10), then inverting $F_k^0(\cdot)$ in the resulting expression, we obtain the identity $v_0(a) \equiv v_k(h_k(a;\theta_0))$.

In practice, of course, θ_0 is unknown. For any conjectured $\theta \in \Theta$, however, we can apply the change of variables $\alpha = h_k(a;\theta)$ in (3) to obtain

$$v_k(h_k(a;\theta)) = b_k(h_k(a;\theta)) + \lambda_0^{-1}(h_k(a;\theta)) \quad \forall a \in [0,1], k \in \mathcal{K}. \tag{11}$$

Moreover, if $\theta = \theta_0$, then $v_0(a) = v_k(h_k(a;\theta))$. Under the hypothesis $\theta = \theta_0$, we can thus link the left-hand sides of (11) across competition levels to obtain a system of *conjectured compatibility conditions* paralleling (7): if $\theta = \theta_0$, then for all $k, l \in \mathcal{K}$ and $a \in [0,1]$,

$$b_k(h_k(a;\theta)) + \lambda_0^{-1}(R_k(h_k(a;\theta))) = b_l(h_l(a;\theta)) + \lambda_0^{-1}(R_l(h_l(a;\theta))). \tag{12}$$

For $\theta \neq \theta_0$, the system (12) will *misspecify* the true bidding relationship, hence there may exist no candidate $\lambda^{-1} \in \Lambda^{-1}$ satisfying (12). By definition, however, $\theta \in \Theta_I$ only if there exists *at least one* such candidate $\lambda^{-1} \in \Lambda^{-1}$. Theorem 2 then further suggests that (12) should determine this λ^{-1} *uniquely* up to a constant on its equilibrium domain $[\underline{r}, \bar{r}]$.

To formalize this intuition, we require one additional regularity condition on $C(a,s;\theta)$. Let $H(a,s;\theta) \equiv \frac{1-\partial C(a,s;\theta)/\partial a}{s+a-C(a,s;\theta)}$, and define $\mathcal{A}_{kl}(\theta) = \left\{a \in [0,1] : \frac{H(a,s_k;\theta)}{H(a,s_l;\theta)} = \frac{N_l-1}{N_k-1}\right\}$ for each $k, l \in \mathcal{K}$ and $\theta \in \Theta$. Each set $\mathcal{A}_{kl}(\theta)$ can be calculated a priori, and as we show in online Appendix **??**, so long as $\mathcal{A}_{kl}(\theta)$ is of measure zero, the bid functions $\beta_k(v), \beta_l(v)$ can intersect on at most a set of measure zero. This in turn allows us to establish the following result:

Theorem 3. *Consider any $\theta \in \Theta_I$ for which there exist at least three distinct competition levels $k, l, m \in \mathcal{K}$ such that the sets $\mathcal{A}_{kl}(\theta)$, $\mathcal{A}_{km}(\theta)$, and $\mathcal{A}_{lm}(\theta)$ are all of measure zero. Then* *there exists a unique function $\phi_{1,\theta} : [\underline{r}, \bar{r}] \to \mathbb{R}^+$, identified up to θ, such that (i) $\phi_{1,\theta}(\underline{r}) = 0$, and (ii) for all $k, l \in \mathcal{K}$ and $a \in [0, 1]$,*

$$b_k(h_k(a;\theta)) + \phi_{1,\theta}(R_k(h_k(a;\theta))) = b_l(h_l(a;\theta)) + \phi_{1,\theta}(R_l(h_l(a;\theta))).$$

Moreover, $\lambda^{-1} \in \Lambda^{-1}$ rationalizes bidding at θ if and only if for some $\phi_2 \in [0, \underline{r}]$,

$$\lambda^{-1}(r) = \phi_{1,\theta}(r) + \phi_2 \quad \forall r \in [\underline{r}, \bar{r}].$$

4 Set inference on bid-stage primitives

We now turn to likelihood-based inference within the AS-RA model. For the sake of practical importance and concreteness, we focus on confidence set constructions for the identified sets of bid-stage primitives $(U_0, F_0, C_0) \in \mathcal{U} \times \mathcal{F} \times \mathcal{C}$ when no cost shifter z is available. We parameterize bid-stage primitives flexibly in terms of Bernstein polynomials as described below. We then extend results in Appendix C of CCT to develop confidence sets for bid-stage primitives using the restrictions in Theorem 1. We note in particular that Theorem C.1 in CCT's Appendix C accommodates set-identified likelihood models with "parameter dependent support" (i.e., support of the observed data depends on the unknown model parameters). This is crucial for our application since, as pointed out by Donald and Paarsch (1993), in first-price auctions every model parameter will typically influence at least one predicted maximum bid. In implementing these methods, we also develop efficient MPEC strategies for solving MLE and profile likelihood problems in first-price auctions, which should be useful in estimation and inference for other complex equilibrium models.

4.1 Flexible parametric likelihood framework

Recall that the bid-stage primitives $(U_0, F_0, C_0) \in \mathcal{U} \times \mathcal{F} \times \mathcal{C}$ consist of smooth functions that also satisfy some shape restrictions and the additional restrictions imposed in Theorem 1. In what follows, we approximate the bid-stage primitives by flexible Bernstein polynomial sieves with large (though fixed) sieve dimensions so that the approximation error (or sieve bias) is of a smaller order and hence could be ignored in first order asymptotics (see, for example, Chen (2007)). We can then interpret bid-stage primitives as belonging to flexible parametric families so that Lemma C.1 and Theorem C.1 of CCT (2018) are applicable.

Since our identification analysis focuses on restrictions on (U_0, F_0, C_0) generated by bidding behavior, we treat entry thresholds $\mathbf{s} \equiv (s_1, ..., s_K)$ as auxiliary parameters to be estimated. Let S denote the admissible set for $\mathbf{s}$: i.e., the set of $\mathbf{s} \in [0, 1]^K$ such that $s_k \geq s_{k-1}$ for all $k = 2, ..., K$. Should one additionally wish to estimate entry costs, one could further enforce the breakeven condition (25) for equilibrium entry described in Appendix A.

We parameterize bid-stage primitives flexibly as follows. Without loss of generality, we re-center bids and values so that $p_0 = 0$. Let $\bar{v} \in [0, \bar{V}]$ and $\underline{v} \in [0, \bar{v}]$, with $\bar{V} < \infty$ a known (but potentially large) upper bound on $\bar{v}$. For any integer $D > 0$ and any $d \in \{0, ..., D\}$, let $B_{d,D}(u)$ denote the dth Bernstein basis polynomial of degree D:

$$B_{d,D}(u) \equiv \binom{D}{d} u^d (1-u)^{D-d}, \quad u \in [0,1].$$

Let $\lambda_0(x) \equiv U_0(x)/U_0'(x)$. We parameterize $\lambda(x) \equiv U(x)/U'(x)$ as a scaled and shifted Bernstein polynomial of degree $Q \geq 1$ with free coefficients $\gamma = (\gamma_j)_{j=1}^Q$.[13] For $x \in [0, \bar{v}]$, we set $\lambda(x) = \bar{v}\tilde{\lambda}(\bar{v}^{-1}x|\gamma)$, where $\tilde{\lambda}(u|\gamma)$ is a shifted degree-Q Bernstein polynomial on $[0, 1]$:

$$\tilde{\lambda}(u|\gamma) \equiv u + \sum_{j=1}^{Q} \gamma_j B_{j,Q}(u), \quad u \in [0,1]. \tag{13}$$

We enforce $\lambda'(x) \geq 1$ by requiring that $\gamma_1 \geq 0$ and $\gamma_j \geq \gamma_{j-1}$ for $j = 2, ..., Q$. We further assume $\gamma_Q \leq \bar{\gamma}$ for some constant $\bar{\gamma} < \infty$.[14] We take γ as parameters to be estimated, belonging to a compact set Γ defined by these inequalities. Recall that if U is CRRA with relative risk aversion ρ, then $\lambda(x) = x/(1-\rho)$. Since Bernstein polynomials nest linearity as a special case, our model thus nests the CRRA model for any $Q \geq 1$. In particular, with $Q = 1$, our model is equivalent to a CRRA model with $\rho = \gamma_1/(1+\gamma_1) \in [0, \bar{\gamma}/(1+\bar{\gamma})]$.

In applications, (F_0, C_0) could be respectively modeled using flexible parameterizations for one- and two-dimensional distributions, such as F_0 by a Bernstein polynomial and C_0 by a Bernstein copula. Recall, however, that conditional on $s_1, ..., s_K$, F_0 and C_0 affect equilibrium bidding behavior only through $F_1^0, ..., F_K^0$ (see equation (1), which with no instruments z becomes $F_k^0(y) = [F_0(y) - C_0(F_0(y), s_k)]/[1 - s_k]$ for all k). We thus instead parameterize $F_1^0, ..., F_K^0$ directly, reinterpreting the conditions of Theorem 1 as constraints on $F_1^0, ..., F_K^0$. Specifically, for each $k = 1, ..., K$, we model F_k as a Bernstein polynomial of degree $P \geq 2$ with free coefficients $\phi_k = (\phi_{k,1}, ..., \phi_{k,P-1})$, scaled to the interval $[\underline{v}, \bar{v}]$:[15]

$$F_k(y) = \tilde{F}_k\left(\frac{y - \underline{v}}{\bar{v} - \underline{v}}\middle|\phi_k\right), \quad \tilde{F}_k(u|\phi_k) \equiv \sum_{j=1}^{P-1} \phi_{k,j} B_{j,P}(u) + B_{P,P}(u), \quad u \in [0,1]. \tag{14}$$

We constrain the coefficients $\phi \equiv (\phi_1, ..., \phi_K)$ to ensure that the implied distributions $F_1, ..., F_K$ satisfy Conditions I-S of Theorem 1. For each $\mathbf{s} \in S$, let the admissible set for ϕ given $\mathbf{s}$, denoted $\Phi(\mathbf{s})$, be the set of vectors ϕ satisfying the following linear inequalities:

[13] The restriction $\lambda(0) = 0$ implies $\gamma_0 = 0$.

[14] This restriction serves to ensure a compact parameter space. A sufficient condition is $RRA(x) < \bar{\gamma}/(1+\bar{\gamma})$ for all $x \geq 0$, where $RRA(x) = -xU''(x)/U'(x)$ denotes the Arrow-Pratt coefficient of relative risk aversion.

[15] The restrictions $F_k(\underline{v}) = 0$ and $F_k(\bar{v}) = 1$ imply $\phi_{k,0} = 0$ and $\phi_{k,P} = 1$ respectively.

M' For each $k = 1, ..., K$, ϕ_k satisfies $0 \le \phi_{k,1} \le \phi_{k,2} \le \cdots \le \phi_{k,P-2} \le \phi_{k,P-1} \le 1$.

O' For all $k = 1, ..., K-1$ and all $u \in [0,1]$,

$$\sum_{j=1}^{P-1} \phi_{k,j} B_{j,P}(u) \ge \sum_{j=1}^{P-1} \phi_{k+1,j} B_{j,P}(u).$$

D' For each $k = 1, ..., K-1$ and all $u \in [0,1]$,

$$(1-s_k)\left(\sum_{j=1}^{P-1} \phi_{k,j} B'_{j,P}(u) + B'_{P,P}(u)\right) \ge (1-s_{k+1})\left(\sum_{j=1}^{P-1} \phi_{k+1,j} B'_{j,P}(u) + B'_{P,P}(u)\right).$$

S' For each $k = 1, ..., K-2$ and all $u \in [0,1]$,

$$\sum_{j=1}^{P-1} \phi_{k,j}\left(\frac{1-s_k}{s_{k+1}-s_k}\right) B_{j,P}(u) + \sum_{j=1}^{P-1} \phi_{k+2,j}\left(\frac{1-s_{k+2}}{s_{k+2}-s_{k+1}}\right) B_{j,P}(u) \\ - \sum_{j=1}^{P-1} \phi_{k+1,j}\left(\frac{1-s_{k+1}}{s_{k+1}-s_k} - \frac{1-s_{k+1}}{s_{k+2}-s_{k+1}}\right) B_{j,P}(u) \ge 0.$$

Conditions O', D', and S' translate Conditions O, D, and S of Theorem 1 into the space of coefficients ϕ used in our parameterizations of $F_1, ..., F_K$; in practice, we enforce these on a fixed grid in $[0,1]$.[16] Meanwhile, Condition M' implies that each F_k is strictly monotone on $[\underline{v}, \bar{v}]$, which in turn implies Conditions I and M of Theorem 1.[17]

Let $\psi \equiv (\bar{v}, \underline{v}, \mathbf{s}, \gamma, \phi) \in \mathbb{R}^{d_\psi}$, with $d_\psi \equiv \dim(\psi) = 2 + K + Q + K(P-1)$, denote the full vector of parameters to be estimated. Given the constraints above, ψ will belong to a known compact set $\Psi \subset \mathbb{R}^{d_\psi}$ defined by

$$\Psi \equiv \left\{\psi = (\bar{v}, \underline{v}, \mathbf{s}, \gamma, \phi) \in \mathbb{R}^{d_\psi} \,|\, \bar{v} \in [0, \bar{V}], \underline{v} \in [0, \bar{v}], \mathbf{s} \in S, \gamma \in \Gamma, \phi \in \Phi(\mathbf{s})\right\} .$$

For any candidate $\psi \in \Psi$ and any competition level $k \in \mathcal{K}$, we let $\beta_k^{-1}(\cdot|\psi)$ denote the inverse of the equilibrium bid function $\beta_k(\cdot|\psi)$ defined in (2) (equivalently (22)), and $\beta_k^{-1,\prime}(b|\psi)$ be its derivative (w.r.t. b). Then the model predicted CDF of equilibrium bids given ψ and k is

$$G_k(b|\psi) = F_k(\beta_k^{-1}(b|\psi)\big|\bar{v}, \underline{v}, \phi_k) \ , \quad \text{for} \quad b \in [\beta_k(\underline{v}|\psi), \beta_k(\bar{v}|\psi)] \ ,$$

[16]For models with $Q > 1$ we enforce O', D', and S' at $u \in \{0.0, 0.1, ..., 0.7, 0.8, 0.85, 0.9, 0.95, .99, 1.0\}$. This spacing emphasizes restrictions for u close to 1, which in view of Theorem 1 we expect to be particularly informative. For point-identified CRRA models with $Q = 1$, we found that enforcing O', D', and S' only for $u \in \{0.95, .99, 1.0\}$ gives very similar results; we thus focus on this simpler grid. This is not surprising in view of Corollary 3, which implies that for $Q = 1$ identification is based primarily on bids near the maximum.

[17]In principle, one could also estimate our model without the auxiliary restrictions M', O', D', S'. A significant improvement in model fit would indicate that the symmetric AS-RA model is misspecified.

with $\beta_k(\underline{v}|\psi) = p_0 \times 1\{s_k \in (0,1)\} + \underline{v} \times 1\{s_k = 0\}$. Letting $f_k(\cdot|\bar{v}, \underline{v}, \phi_k)$ be the density implied by $F_k(\cdot|\bar{v}, \underline{v}, \phi_k)$, the predicted density of equilibrium bids $g_k(\cdot|\psi)$ is given by

$$g_k(b|\psi) \equiv f_k(\beta_k^{-1}(b|\psi)|\bar{v}, \underline{v}, \phi_k) \times \beta_k^{-1,\prime}(b|\psi) \times 1\{\beta_k(\underline{v}|\psi) \le b \le \beta_k(\bar{v}|\psi)\} \ . \tag{15}$$

Unfortunately, for any given parameter $\psi \in \Psi$, the function $b \mapsto \beta_k^{-1}(b|\psi)$ does not have an analytic form in general. We present another expression for the bid density $g_k(b|\psi)$ in (23) below, which is what we use in likelihood evaluation in practice.

We consider an i.i.d. sample of L auctions with public reserve price p_0, where N_k varies exogenously on the set $\mathcal{N} = \{N_1, ..., N_K\}$ but no cost shifter z is available. For each auction $l = 1, ..., L$, we observe the competition level $k_l \in \mathcal{K}$, the entry decision $e_{il} \in \{0,1\}$ for each potential bidder $i = 1, ..., N_{k_l}$, and the bid b_{il} submitted by each entrant. The conditional log-likelihood of observing outcome (e_{il}, b_{il}) in auction l given N_{k_l} and parameters $\psi \in \Psi$ is

$$\ell_{il}(\psi) = \log\left(s_{k_l}^{(1-e_{il})}\left[(1-s_{k_l})g_{k_l}(b_{il}|\psi)\right]^{e_{il}}\right) \quad \forall k_l. \tag{16}$$

Assuming $s_{k_l} \in (0,1)$, (16) is well-defined and finite for all $\psi \in \Psi$ satisfying the upper support constraint $\beta_{k_l}(\bar{v}|\psi) \ge \hat{\bar{b}}_{k_l}$, where $\hat{\bar{b}}_k$ denotes the maximum bid observed at competition level k, which is the maximum order statistic of bids observed at competition level k. (This is a super consistent estimator of the true upper support $\bar{b}_k^0$ of population equilibrium bids at competition level k, which satisfies $\hat{\bar{b}}_k \le \bar{b}_k^0$ in each finite sample.)

Finally, let $\mathcal{L}(\psi)$ denote the sample log-likelihood derived from (16), assuming a random sample $\{(e_{il}, b_{il})_{i=1}^{N_{k_l}}\}_{l=1}^{L}$ from L independent auctions:

$$\mathcal{L}(\psi) \equiv \sum_{l=1}^{L}\sum_{i=1}^{N_{k_l}} \ell_{il}(\psi). \tag{17}$$

Let $\Psi_I \equiv \arg\max_{\psi \in \Psi} \mathbb{E}[\ell_{il}(\psi)]$ be the identified set for the parameter vector ψ. We let $\psi_0 \in \Psi_I$ wlog. Let $\hat{\psi} \in \Psi$ be an approximate maximum likelihood estimator (MLE), i.e.,

$$\mathcal{L}(\hat{\psi}) = \max_{\psi \in \Psi} \mathcal{L}(\psi) + o_p(1).$$

The log-likelihood ratio (LR) statistic is defined as $\mathcal{Q}(\psi) \equiv 2[\mathcal{L}(\hat{\psi}) - \mathcal{L}(\psi)]$. We aim to conduct inference based on $\mathcal{Q}(\psi)$ without assuming point identification of ψ.

4.2 Confidence sets for the identified set of ψ

Let $\alpha \in (0,1)$ be any target confidence level, say $\alpha = 0.95$. We aim to construct a $100\alpha\%$ confidence set (CS) $\hat{\Psi}_\alpha$ for the identified set Ψ_I, with asymptotically exact coverage in the sense that $\lim_n Pr\left(\Psi_I \subseteq \hat{\Psi}_\alpha\right) = \alpha$ or conservative coverage $\lim_n Pr\left(\Psi_I \subseteq \hat{\Psi}_\alpha\right) \ge \alpha$.

Toward this end, let Π be a continuous prior that puts positive weight over Ψ. Given this prior and the data $\{(e_{il}, b_{il})_{i=1}^{N_l}\}_{l=1}^{L}$, the posterior distribution $\Pi_{\mathcal{L}}$ for ψ is

$$d\Pi_{\mathcal{L}}(\psi) \equiv \frac{\exp[\mathcal{L}(\psi)]\, d\Pi(\psi)}{\int_{\Psi} \exp[\mathcal{L}(\psi)]\, d\Pi(\psi)} = \frac{\exp[-0.5\mathcal{Q}(\psi)]\, d\Pi(\psi)}{\int_{\Psi} \exp[-0.5\mathcal{Q}(\psi)]\, d\Pi(\psi)}. \tag{18}$$

Procedure 1 of CCT constructs a CS for Ψ_I as follows: first draw a sample of parameters $\{\psi^b\}_{b=1}^{B^*}$ from $\Pi_{\mathcal{L}}$, then calculate a critical value ξ_α as the αth quantile of $\{\mathcal{Q}(\psi^b)\}_{b=1}^{B^*}$. The set $\hat{\Psi}_\alpha = \{\psi \in \Psi : \mathcal{Q}(\psi) \leq \xi_\alpha\}$ is then a $100\alpha\%$ CS for Ψ_I.

In their Lemma C.1, CCT establish a generalized Bernstein-von Mises theorem applicable under set identification and with or without parameter dependent support. Under some sufficient conditions that are restated in our Appendix B, this result implies that both the distribution of $\mathcal{Q}(\psi)$ when ψ is sampled from the posterior $\Pi_{\mathcal{L}}$ and the distribution of the frequentist LR statistic $\mathcal{Q}(\psi_0)$ across samples will converge to a common asymptotic distribution of the form Gamma$(r^*, 2)$, which is a Gamma distribution with shape parameter $r^* > 0$ and scale parameter 2, although r^* is typically unknown for set-identified models.[18] Consequently, the αth quantiles of posterior and frequentist LR statistics coincide asymptotically. This in turn justifies our application of CCT's Procedure 1 for asymptotically exact frequentiest CSs for Ψ_I in point or set-identified auction models.

4.3 Confidence sets for identified sets of subvectors of ψ

Let $\eta(\psi)$ be any subvector (or functional) of $\psi \in \Psi$, and $\mathcal{M}_I := \{\eta(\psi) : \psi \in \Psi_I\}$ be the bid-stage identified set for $\eta(\psi)$. The projection CS $\hat{M}_\alpha^{proj} = \{\eta(\psi) : \psi \in \hat{\Psi}_\alpha\}$ for $\mathcal{M}_I$ is valid whenever $\hat{\Psi}_\alpha$ is a valid $100\alpha\%$ CS for Ψ_I. However, the projection CS $\hat{M}_\alpha^{proj}$ is typically very conservative in the sense of $\lim_n Pr\left(\mathcal{M}_I \subseteq \hat{M}_\alpha^{proj}\right) >> \alpha$ for $\eta(\psi)$ when $\eta(\psi)$ is a low dimensional subvector (say a scalar) of ψ. We instead apply CCT's Procedure 2 to construct CSs for $\mathcal{M}_I$, extending their Lemma C.1 to show that their Procedure 2 remains valid for subvector inference for models with parameter dependent support.

In generic likelihood models, Procedure 2 can be implemented as follows. For each value η_1 in the domain of $\eta(\psi)$, let $\tilde{\mathcal{L}}(\eta_1) = \sup_{\psi\in\Psi}\{\mathcal{L}(\psi) : \eta(\psi) = \eta_1\}$ denote the profiled log-likelihood fixing $\eta(\psi) = \eta_1$. For $\psi, \psi' \in \Psi$, let $D_{KL}(\psi||\psi')$ be Kullback-Liebler (KL) diver-

[18] It is well-known that for a point-identified model $\Psi_I = \{\psi_0\}$ belonging to the interior of Ψ, the frequentist LR statistic $\mathcal{Q}(\psi_0)$ is asymptotically chi-square χ^2_ν distributed with degree of freedom $\nu = \dim(\psi_0)$, which is Gamma$(\nu/2, 2)$ if and only if $r^* = \dim(\psi_0)/2$. For a point-identified AS-RA auction model with multi competition levels (i.e., $K > 1$), the shape parameter r^* will depend on difficult-to-characterize interactions between parameters and support constraints $\beta_k(\bar{v}_0|\psi_0) = \bar{b}_k^0 \geq \hat{\bar{b}}_k$, especially when the equilibrium bid functions $\beta_k(\cdot|\psi)$, $k = 1, ..., K$, have no closed-form expressions.

gence from ψ to ψ', which for the log likelihood (16) can be expressed as

$$D_{KL}(\psi||\psi') = \sum_{k=1}^{K} \left[s_k \log\left(\frac{s_k}{s'_k}\right) + (1-s_k)\int_B \log\left(\frac{(1-s_k)g_k(B|\psi)}{(1-s'_k)g_k(B|\psi')}\right) dG_k(B|\psi)\right].$$

Let $\{\psi^b\}_{b=1}^{B^*}$ be a sample of parameters drawn from $\Pi_{\mathcal{L}}$ as in Procedure 1 above. For each parameter draw ψ^b, define a profile criterion $PL(\psi^b)$ as follows:

$$PL(\psi^b) = \inf\{\tilde{\mathcal{L}}(\eta') : \eta' = \eta(\psi') \text{ for } \psi' \in \Psi \text{ with } D_{KL}(\psi^b||\psi') = 0\}. \tag{19}$$

Finally, let $\zeta_{1,\alpha}$ be the $(1-\alpha)$th quantile of the sample $\{PL(\psi^b)\}_{b=1}^{B^*}$. As we show in Theorem 5 in Appendix B, under some sufficient conditions, the set $\hat{M}_\alpha = \{\eta : \tilde{\mathcal{L}}(\eta) \geq \zeta_{1,\alpha}\}$ is then a $100\alpha\%$ CS for $\mathcal{M}_I$.

We simplify this implementation by replacing the generic profile criterion (19) with the following alternative, which leverages our analytic characterization of identified sets to speed up computation substantially. By Theorem 1, fixing $\mathbf{s} = \mathbf{s}'$, we have $G_k(\cdot|\psi) = G_k(\cdot|\psi')$ if and only if $v_k(a|\psi') = b_k(a|\psi) + \lambda^{-1}(R_k(a|\psi)|\psi')$ for all $a \in [0,1]$. Motivated by this fact, we define a L^2-type directed distance function $D_{L^2}(\psi||\psi')$ as follows:

$$D_{L^2}(\psi||\psi') = \sum_{k=1}^{K} \int_0^1 \left[F_k\Big(b_k(a|\psi) + \lambda^{-1}\big(R_k(a|\psi)\big|\psi'\big)\Big|\psi'\Big) - a\right]^2 da. \tag{20}$$

Fixing $\mathbf{s} = \mathbf{s}'$, we can then have $D_{KL}(\psi||\psi') = 0$ if and only if $D_{L^2}(\psi||\psi') = 0$. We thus obtain an equivalent, but computationally more convenient, definition of $PL(\psi^b)$:

$$PL(\psi^b) = \inf\{\tilde{\mathcal{L}}(\eta') : \eta' = \eta(\psi') \text{ for } \psi' \in \Psi \text{ with } \mathbf{s}^b = \mathbf{s}' \text{ and } D_{L^2}(\psi^b||\psi') = 0\}. \tag{21}$$

To trace out $D_{L^2}(\psi||\psi')$ as a function of ψ', one need only re-evaluate $F_k(\cdot|\psi')$ and $\lambda^{-1}(\cdot|\psi')$ holding $b_k(a|\psi)$ and $R_k(a|\psi)$ fixed, whereas to trace out $D_{KL}(\psi||\psi')$ one must re-solve equilibrium bid functions anew at every guess of ψ'. In our simulations, evaluation of $PL(\psi^b)$ based on (21) is thus between one and two orders of magnitude faster than that based on (19). This finding highlights how identified sets based on equilibrium conditions, such as provided by our Theorem 1, can be applied to simplify finite-sample inference.

4.4 Inference for point identified bid-stage primitives

CCT's Procedures 1 and 2 are valid regardless of whether a model is set identified or point identified. Recall, however, that in our setting $Q = 1$ corresponds to CRRA utility, which implies point identification of ψ_0 by Corollary 3.[19] Under point identification, although Procedure 1 is implemented the same way, Procedure 2 further simplifies to using the standard

[19] Note that ψ_0 parameterizes λ_0 and $F_1^0, ..., F_K^0$, which are point identified under CRRA U_0 by Corollary 3. If one instead flexibly parameterized (F_0, C_0), then ψ_0 may not be point identified, but the simplifications noted here would still apply to any functional which depends on ψ_0 only through λ_0 and / or $F_1^0, ..., F_K^0$.

profiled likelihood criterion $PL(\psi^b) = \tilde{\mathcal{L}}(\eta(\psi^b))$, with no need to numerically evaluate the infimum of $\tilde{\mathcal{L}}(\eta')$ over an ex ante unknown equivalent set. This dramatically simplifies subvector inference based on Procedure 2, even relative to (21). In practice, we recommend starting with the point-identified CRRA model $Q = 1$, a specification of interest in its own right which also nests risk neutrality as a special case. If estimates with $Q = 1$ indicate risk aversion, one can then proceed to more flexible models with $Q > 1$.

4.5 Computational implementation

In implementing the methods above, we have developed fast and stable strategies for solving MLE and profiled likelihood problems, which are particularly valuable when closed-form equilibrium bid functions are not available. We briefly outline the main elements of our novel strategies here; see Online Appendix **??** for details within our simulation study in Section 5. First, for any given parameter $\psi \in \Psi$ and any competition level $k \in \mathcal{K}$, rather than solving for equilibrium bid functions $\beta_k(\cdot|\psi)$ directly using the differential equation (DE) (2)

$$\beta_k'(v|\psi) = \lambda(v - \beta_k(v|\psi)\big|\bar{v}, \gamma)\frac{(N_k - 1)(1 - s_k)f_k(v|\bar{v}, \underline{v}, \phi_k)}{s_k + (1 - s_k)F_k(v|\bar{v}, \underline{v}, \phi_k)}, \tag{22}$$

with $\beta_k(\underline{v}|\psi) = p_0 \times 1\{s_k \in (0,1)\} + \underline{v} \times 1\{s_k = 0\}$. We re-express this as a system of Chebyshev collocation equations and enforce these as constraints in MPEC optimization. Second, rather than evaluate the bid density $g_k(b_{il}|\psi)$ in (15) directly, we substitute from the DE (22) to obtain a simpler and more stable expression valid on its support $[\beta_k(\underline{v}|\psi),\ \beta_k(\bar{v}|\psi)]$:

$$g_k(b_{il}|\psi) = \frac{f_k(v_{il}|\bar{v}, \underline{v}, \phi_k)}{\beta_k'(v_{il}|\psi)} = \frac{s_k + (1 - s_k)F_k(v_{il}|\bar{v}, \underline{v}, \phi_k)}{\lambda(v_{il} - b_{il}\big|\bar{v}, \gamma)(N_k - 1)(1 - s_k)}, \tag{23}$$

where v_{il} is defined implicitly by $b_{il} \equiv \beta_k(v_{il}|\psi)$. Third, to further streamline computation, we evaluate (23) exactly for bids on an interval near the maximum bid at each competition level (which our theory suggests are pivotal for identification) while discretizing bids below this interval on a fine grid. In our simulations, this partial discretization scheme substantially improves both speed (requiring far fewer numerical bid inversions) and stability (by eliminating potential numerical singularities induced by the fact that when $p_0 = \underline{v}$, $\log g_k(b\big|\psi) \to \infty$ as $b \to p_0$), with minimal impact on precision.[20] We thus obtain a fast and stable algorithm for solving MLE problems in first-price auctions; in our simulations, the median such problem is solved in between one and three seconds, even in models with more than 30 parameters.

[20]We experimented in detail with approximation grids and regions, finding that for the grids we consider, which are much finer than the degree of our polynomial approximations, even large changes in the discretization region had negligible impacts on profiled Kullback-Leibler loss and profiled likelihood estimates.

5 A simulation exercise

Finally, we explore our proposed inference methods in a simulation study based on the following true data generating process (DGP). Bidders have CRRA utility: $U_0(x) = U(x; \rho_0) = x^{1-\rho_0}$ or equivalently $\lambda_0(x) \equiv U_0(x)/U_0'(x) = x/(1-\rho_0)$. Valuations are drawn from a truncated logistic distribution F_0 with mean 0.5 and scale 0.3, truncated on the interval $[\underline{v}_0, \bar{v}_0]$ with $\underline{v}_0 = 0$ and $\bar{v}_0 = 1$. The reserve price is $p_0 = 0$. For simplicity, and for this section only, we assume the econometrician knows the reserve price is just-binding, so that $\underline{v}_0 = p_0 = 0$.[21] Dependence between V_i and S_i is parameterized by a Frank copula

$$C(F_0, s; \theta) = -\frac{1}{\theta} \log \left[1 + \frac{(\exp(-\theta F_0) - 1)(\exp(-\theta s) - 1)}{\exp(-\theta) - 1} \right] \quad \text{for} \quad 0 < \theta < \infty,$$

with true parameter value $\theta_0 = 2.0$, corresponding to a Spearman's rank correlation between V_i and S_i of approximately 0.32. The entry cost is $c_0 = 0.1$. We consider $K = 4$ competition levels, with N varying exogenously over the values $N_k \in \{2, 4, 6, 8\}$. We consider both risk averse ($\rho_0 = 0.5$) and risk neutral ($\rho_0 = 0$) DGPs. Entry thresholds are approximately $s_k \in \{0.16, 0.52, 0.66, 0.74\}$ for our main DGP with $\rho_0 = 0.5$, and $s_k \in \{0.00, 0.09, 0.28, 0.40\}$ for our risk-neutral DGP with $\rho_0 = 0$. The true $F_1^0, ..., F_K^0$ can be solved from $F_0, C(F_0, s; \theta_0)$ through (1), i.e., $F_k^0(y) = [F_0(y) - C(F_0(y), s_k; \theta_0)]/[1 - s_k]$ for all k. For each DGP, we consider two sample scales $M \in \{1000, 2000\}$, where M represents the approximate average number of bids observed per competition level $k = 1, ..., 4$. For each Monte Carlo replication, we simulate $L_k = \lceil \frac{M}{N_k(1-s_k)} \rceil$ auctions at each $k = 1, ..., 4$, for a total of $L = \sum_{k=1}^{4} L_k$ auctions.

We consider two econometric model specifications, which we label the *Flexible copula* and *Frank copula* models respectively. For both models, we set $\underline{v} = 0$ and parameterize the unknown true $\lambda_0(x)$ as a Bernstein polynomial (with parameters γ) of either degree $Q = 1$ (the true CRRA model), $Q = 4$, or $Q = 6$.

- For the *Flexible copula* model, we leave the copula unspecified, and approximate true $F_1^0, ..., F_K^0$ flexibly using Bernstein polynomials (with parameters ϕ) of degree $P \in \{4, 5, 6\}$ as described in Section 4.1. The implied bid density $g_k(b_{il}|\psi)$ is given in (23), with unknown parameter $\psi = (\bar{v}, \mathbf{s}, \gamma, \phi)$ of dimension $d_\psi = 1 + 4 + Q + 4(P - 1)$.

- For the *Frank copula* model, we correctly specify the copula as a parametric Frank copula up to unknown $\theta \in \mathbb{R}$, and approximate the true F_0 by a Bernstein polynomial (with parameters ϕ_o) of degree $P \in \{4, 5, 6\}$. The corresponding bid density $g_k(b_{il}|\psi)$ is given

[21] Our flexible parameterizations for $F_1^0, ..., F_K^0$ do, however, allow for densities which are close to zero near $\underline{v}_0 = 0$, which although distinct in terms of identification closely approximate varying $\underline{v}$ in finite samples.

in (23) except that $(1-s_k)F_k(v_{il}|\bar{v},\phi_k)$ is replaced by $[F(v_{il}|\bar{v},\phi_o) - C(F(v_{il}|\bar{v},\phi_o), s_k;\theta)]$, with unknown parameter $\psi = (\bar{v}, \mathbf{s}, \gamma, \theta, \phi_o)$ of dimension $d_\psi = 1 + 4 + Q + 1 + (P-1)$.

Note that, by design, both econometric models are only approximately correct in the sense that the true DGPs $F_1^0, ..., F_K^0$ and F_0 are not nested by our parametric Bernstein polynomials of fixed finite degree $P \in \{4, 5, 6\}$, but (as we show in the next subsection) the Kullback-Leibler divergence between the true DGPs and our econometric models are very small. We have made such choices to illustrate that, although we do not study asymptotics with growing numbers of parameters rigorously, our inference methods work well in practice even when exact true parametric forms are not necessarily known.

For concreteness, we focus inference on the average slope $\tilde{\lambda}(x|\gamma)/x$ of $\tilde{\lambda}(x|\gamma)$ defined in (13), which, for fixed $x \in [0, 1]$, is a linear functional of γ (and thus ψ). The true value of this average slope is $\bar{\lambda}_0 = \tilde{\lambda}(x|\gamma_0)/x = 1/(1-\rho_0)$ for all $x \in [0, 1]$, which is a measure of risk attitudes, with larger values of $\bar{\lambda}_0$ indicating greater departures from risk neutrality.

5.1 Well-approximated likelihood, identified set, posterior LR

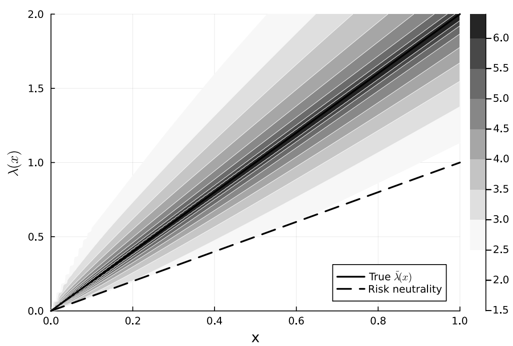
We first numerically explore approximation accuracy in our Flexible copula model. Let $D_{KL}^0(\psi)$ denote KL divergence from a given true DGP to our model at parameters $\psi \in \Psi$, normalizing units of D_{KL}^0 such that $M \cdot D_{KL}^0(\psi)$ represents expected log-likelihood loss in a sample of scale M.[22] Let $\hat{D}_{KL}^0 = \min_{\psi\in\Psi} D_{KL}^0(\psi)$ be minimum KL divergence (model entropy) relative to the true DGP. Comparing Flexible copula models with $P \in \{4, 5, 6\}$, we find that $\hat{D}_{KL}^0$ is on the order of 10^{-4} for $P = 4$, 10^{-5} for $P = 5$, and 10^{-7} for $P = 6$—the latter sufficiently small that $P = 6$ can be taken "as if" the true DGP. Even at $P = 4$, however, losses due to approximation are small both absolutely and relative to penalties on P under model selection rules such as the Akaike Information Criterion (AIC). Consequently, with $Q = 1$ at scale $M = 2000$, AIC selects $P = 4$ among $P \in \{4, 5, 6\}$ in 96.7 percent (87.6 percent) of simulations in our risk-averse DGP (risk-neutral DGP) respectively.[23]

Second, we numerically explore identification under CRRA ($Q = 1$) and flexible ($Q > 1$) parameterizations of unknown true $\lambda_0(x)$ in our Flexible copula model, focusing on specifications with $P = 6$ where approximation error is negligible ($\hat{D}_{KL}^0 \approx 10^{-7}$). For any functional η of ψ and any conjectured value η_1 of η, let $pKL(\eta_1) = \min_{\psi'\in\Psi}\{D_{KL}^0(\psi') : \eta(\psi) = \eta_1\}$ denote profiled KL divergence fixing $\eta(\psi) = \eta_1$. Figure 2 plots, for models with $P = 6$ and $Q \in \{1, 4, 6\}$, contours of pKL for $\tilde{\lambda}(x|\gamma)$, interpreting $\tilde{\lambda}(x|\gamma)$ pointwise for each $x \in [0, 1]$ as

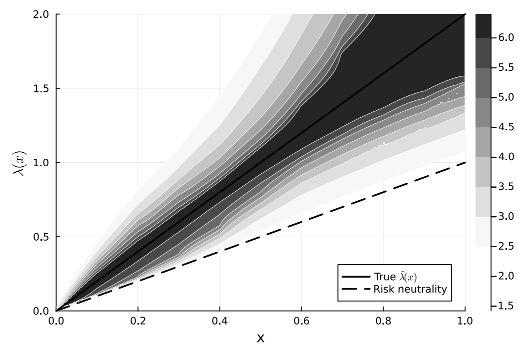
[22] To ensure internal consistency, we implement D_{KL}^0 and $D_{KL}(\psi||\psi')$ using the same hybrid of grid and exact evaluation as our numerical log-likelihood implementation, as described in online Appendix **??**.

[23] Comparing models with $Q \in \{1, 4, 6\}$ and $P = 6$ at $M = 2000$, we also find that $Q = 1$ is preferred in all simulations, as expected since our DGP involves $Q = 1$.

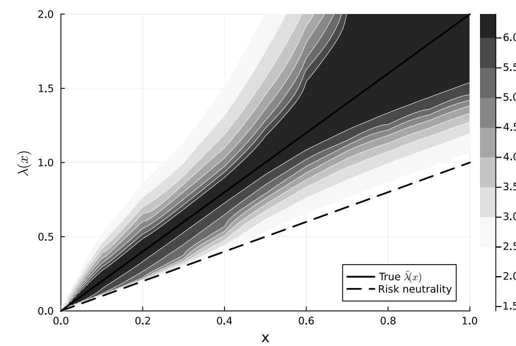
Figure 2: Pointwise Kullback-Leibler contour sets for $\tilde{\lambda}(x|\gamma_0) = x/(1-\rho_0)$, parameterizing $\lambda_0(x)$ as Bernstein of order $Q \in \{1, 4, 6\}$ and $F_1^0, ..., F_K^0$ as Bernstein of order $P = 6$.



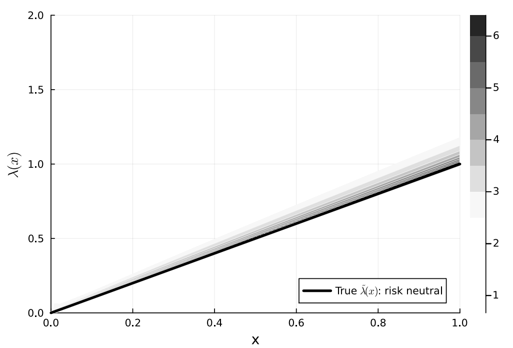
(a) $\rho = 0.5$, $Q = 1$



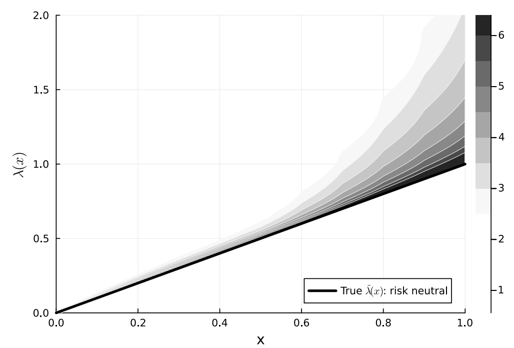
(b) $\rho = 0.5$, $Q = 4$



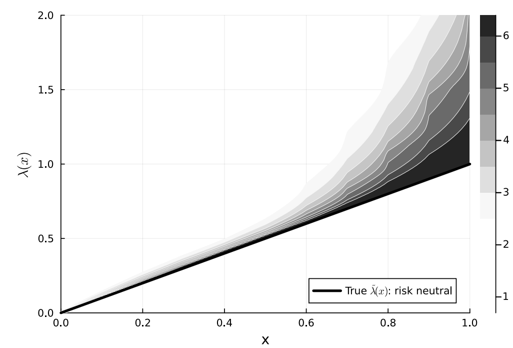
(c) $\rho = 0.5$, $Q = 6$



(d) $\rho = 0.0$, $Q = 1$



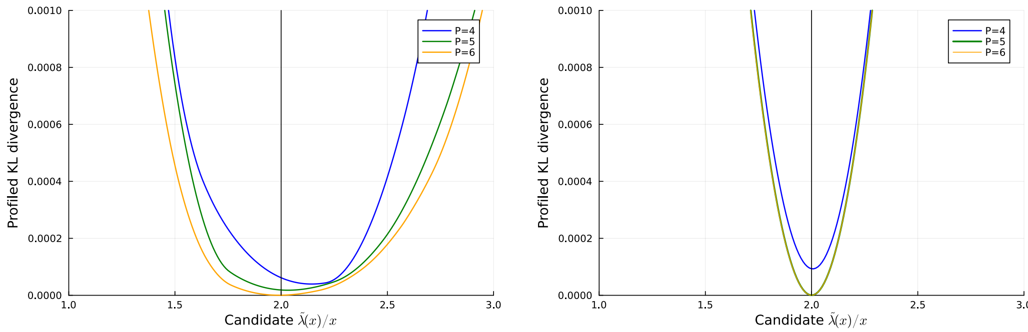
(e) $\rho = 0.0$, $Q = 4$



(f) $\rho = 0.0$, $Q = 6$

Notes: Contour values reflect the inverse order of magnitude of the pointwise profiled KL divergence $\inf_{\psi \in \Psi}\{D_{KL}^0(\psi) : \tilde{\lambda}(x) = y\}$. For example, a contour value above 5.0 implies that there is a parameter vector $\psi \in \Psi$ with $\tilde{\lambda}(x) = y$ such that $D_{KL}^0(\psi) \leq 10^{-5.0}$.

Figure 3: Profiled Kullback-Leibler divergence for $\bar{\lambda}_0 = 1/(1-\rho_0)$ in risk-averse CRRA models ($\rho_0 = 0.5$, $Q = 1$) based on Frank copula versus Flexible copula specifications.



(a) Frank: F_0 Bernstein-P, C Frank (b) Flexible: $F_1, \dots, F_K$ Bernstein-P

a functional of γ.[24] For both DGPs, the CRRA model $Q = 1$ is clearly point identified. By contrast, especially for our risk-averse DGP, contour sets for $Q \in \{4, 6\}$ are numerically flat around $\tilde{\lambda}(x|\gamma_0)$, consistent with set identification and fundamentally different from the point identified model $Q = 1$. Both patterns are expected in view of Theorem 1 and Corollary 3.

Third, focusing on the point-identified CRRA case ($Q = 1$), we explore how our main Flexible copula model compares to our benchmark Frank copula model. Specifically, Figure 3 compares pKL for $\bar{\lambda}_0 = \tilde{\lambda}(x|\gamma_0)/x$ in Frank copula models with $Q = 1$ and $P \in \{4, 5, 6\}$ (Panel (a)) to that in Flexible copula models with $Q = 1$ and $P \in \{4, 5, 6\}$ (Panel (b)). Contours of pKL in the Frank copula model are steep and nearly quadratic, whereas those in the Flexible copula model are much flatter near $\bar{\lambda}_0$. These patterns reflect the fact that identification is based on all bid quantiles in the Frank copula model, but primarily on bids near the maximum bid at each $k \in \mathcal{K}$ in the Flexible copula model. They also lead us to expect that profiled LR CSs for $\bar{\lambda}_0$ in the Flexible copula model will tend to be conservative in finite samples, only approaching correct asymptotic size in very large samples.

Finally, we provide direct simulation evidence for the key property underpinning frequentist validity of our CSs: both posterior and frequentist LR statistics converge to a common Gamma$(r^*, 2)$ distribution for some $r^* > 0$. Toward this end, in Figure 4, we compare distributions of posterior and frequentist LR statistics for both risk-neutral ($\rho_0 = 0.5$) and risk-averse ($\rho_0 = 0.0$) DGPs.[25] We consider four models at $M = 2000$: a point-identified

[24] Even abstracting from approximation issues, it is not feasible to compute either the identified set or its projections exactly when $Q > 1$, since to compute $D_{KL}(\psi_0||\psi)$ we must first solve the functional equation (2) defining equilibrium bidding functions $\beta_k(\cdot|\psi)$. For ϵ of greater order than approximation error, contours of D^0_{KL} and their projections will represent outer bounds on true (or pseudo-true) identified sets.

[25] In field data, where true DGPs are unknown, one could conduct similar validation exercises using data simulated from MLE parameter estimates or selected posterior sampling draws.

Frank copula model with CRRA utility ($Q = 1$) and $P = 5$ (row 1), point-identified Flexible copula models with CRRA utility ($Q = 1$) and $P = 4$ or $P = 6$ (rows 2 and 3), and a partially identified Flexible copula model with $Q = 4$ and $P = 6$ (row 4). For each model and DGP, we fit a Gamma$(r^*, 2)$ distribution to the sample of posterior LR statistics $\mathcal{Q}(\psi)$ obtained from a uniform prior (pooling draws $\{\mathcal{Q}(\psi^b)\}_{b=1}^{B^*}$ across simulations). We then plot quantiles of this Gamma$(r^*, 2)$ distribution against those of (i) the simulated distribution of posterior LR statistics $\mathcal{Q}(\psi)$, and (ii) the simulated distribution of the frequentist LR statistic $\mathcal{Q}(\psi_0)$ across replications. Each resulting scatter plot lies on or very close to the relevant diagonal, confirming that each pair of posterior and frequentist LR distributions is indeed well approximated by a common Gamma$(r^*, 2)$ distribution. As such, Figure 4 strongly supports frequentist validity of CCT's CSs in our first-price auction setting.

5.2 Inference results

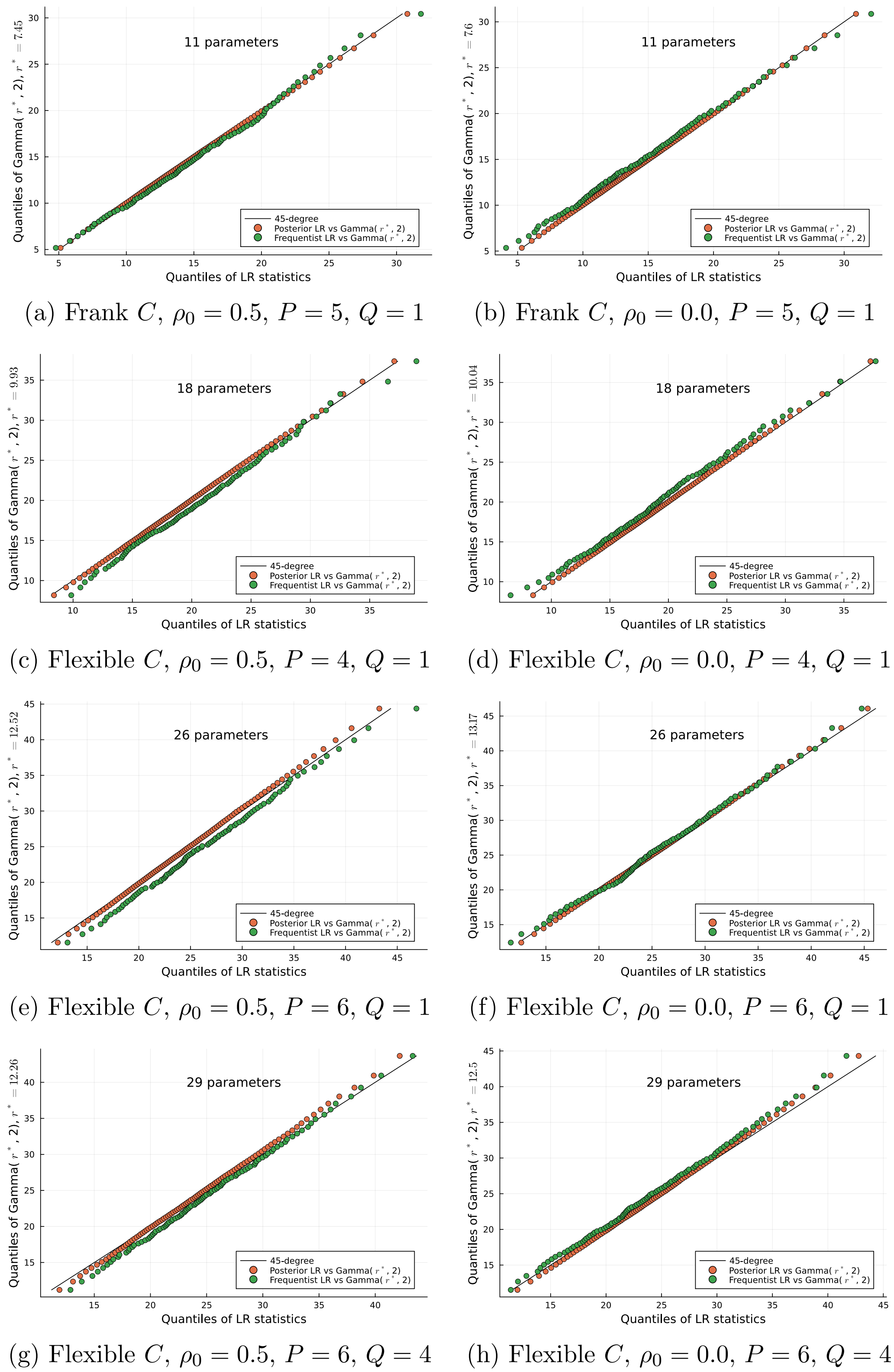
In implementing our CSs, we explore both flat and "boundary" priors Π over Ψ.[26] Specifically, for scalars $\Delta_\gamma, \Delta_\phi > 0$, our boundary priors set $\log \Pi_\gamma(\gamma) \propto -\sum_{j=1}^{Q} \log(\gamma_j - \gamma_{j-1} + \Delta_\gamma)$ and $\log \Pi_{\phi_k}(\phi_k) \propto -\sum_{j=1}^{P} \log(\phi_{k,j} - \phi_{k,j-1} + \Delta_\phi)$, with flat priors on other parameters. Since Ψ is a compact set, this defines a proper prior for any $\Delta_\gamma, \Delta_\phi > 0$. As $\Delta_\gamma, \Delta_\phi \to 0$, greater prior weight is placed on parameter values near monotonicity constraints, while as $\Delta_\gamma, \Delta_\phi \to \infty$, the prior approaches a uniform distribution over Ψ. We draw parameters $\{\psi^b\}_{b=1}^{B^*}$ from $\Pi_{\mathcal{L}}$ using a hybrid of Hamiltonian Monte Carlo (HMC) and Metropolis-Hastings Markov Chain Monte Carlo (MH-MCMC) algorithms described in Online Appendix **??**.

5.2.1 Inference for point-identified models ($Q = 1$)

We first explore inference within the point-identified CRRA model ($Q = 1$), for which γ is a scalar and $\bar{\lambda}_0 = \tilde{\lambda}(x|\gamma_0)/x = (1 + \gamma_0) = 1/(1 - \rho_0)$ for all $x \in [0, 1]$. We compare CSs for $\bar{\lambda}_0$ based on CCT's Procedure 2 with two alternative methods valid under point identification: CSs based on percentiles of Monte Carlo (MC) parameter draws $\{\gamma^b\}_{b=1}^{B}$, which are standard in the Bayesian literature and valid for interior (but not boundary) scalar parameter that is asymptotically root-L normally distributed, and CSs derived from inversion of the standard profiled LR test statistic $\tilde{\mathcal{Q}}(\gamma) = 2(\mathcal{L}(\hat{\psi}) - \tilde{\mathcal{L}}(\gamma))$. For the standard profiled LR inference, we

[26] Especially when P and Q are large, flat priors may lead to poor exploration of "corners" of Ψ where multiple monotonicity constraints bind. For example, a uniform prior over the feasible set Γ for γ implies that $\tilde{\lambda}(1|\gamma)$ has the same distribution as the highest of Q draws from a uniform distribution over $[1, \bar{\gamma} + 1]$, which for $Q > 1$ assigns probability approaching zero to values near the risk-neutral boundary $\tilde{\lambda}(1|\gamma) = 1$. See Ghosal and van der Vaart (2017) for a textbook discussion of sieve priors in nonparametric Bayesian inference.

Figure 4: Q-Q plots of simulated posterior and frequentist LR quantiles.



(a) Frank C, $\rho_0 = 0.5$, $P = 5$, $Q = 1$

(b) Frank C, $\rho_0 = 0.0$, $P = 5$, $Q = 1$

(c) Flexible C, $\rho_0 = 0.5$, $P = 4$, $Q = 1$

(d) Flexible C, $\rho_0 = 0.0$, $P = 4$, $Q = 1$

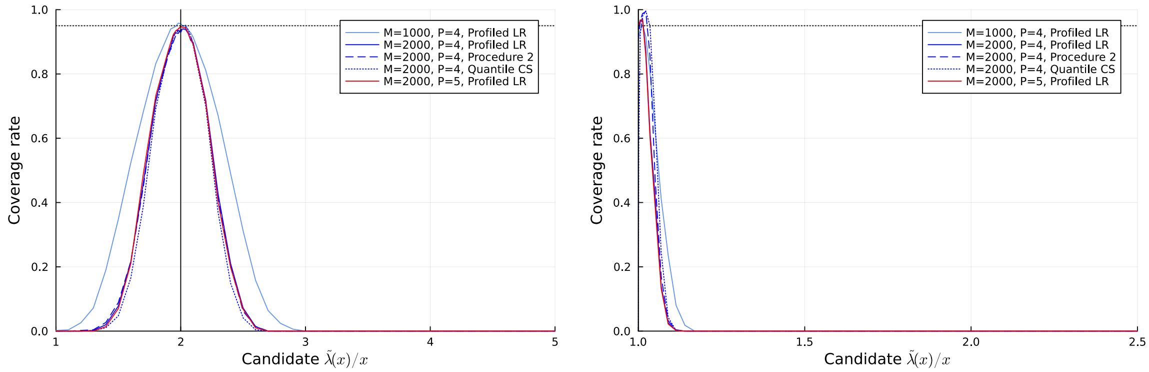
(e) Flexible C, $\rho_0 = 0.5$, $P = 6$, $Q = 1$

(f) Flexible C, $\rho_0 = 0.0$, $P = 6$, $Q = 1$

(g) Flexible C, $\rho_0 = 0.5$, $P = 6$, $Q = 4$

(h) Flexible C, $\rho_0 = 0.0$, $P = 6$, $Q = 4$

Figure 5: **Frank copula,** $Q = 1$**:** Coverage curves based on alternative CSs for $\bar{\lambda}_0$, parameterizing $\lambda_0(\cdot)$ as CRRA ($Q = 1$), C as Frank, and F_0 as Bernstein of order $P \in \{4, 5\}$.



(a) Risk averse DGP: $\rho_0 = 0.5$

(b) Risk neutral DGP: $\rho_0 = 0.0$

Notes: Estimates based on 500 replications, MC CSs use $B^* = 400$ draws. M is the average number of bids observed per competition level. Vertical lines denote true $\tilde{\lambda}(x|\gamma_0)/x = \bar{\lambda}_0 = 1/(1 - \rho_0)$.

form $100\alpha\%$ CSs for $\bar{\lambda}_0$ by $CS_\alpha^{LR} = \{\tilde{\lambda}(x|\gamma)/x : \gamma \in [0, \bar{\gamma}], \tilde{\mathcal{Q}}(\gamma) \leq \xi_\alpha^{LR}\}$, where the critical value ξ_α^{LR} equals the αth quantile of the chi-square χ_1^2 (respectively chi-bar $\bar{\chi}_1^2$) distribution for our interior $\rho_0 = 0.5$ (respectively boundary $\rho_0 = 0.0$) DGP.[27]

We begin with our benchmark Frank copula model, for which we implement MC CSs (Procedure 2 and percentile CSs) using simple flat priors. In view of Figure 3, we expect that results from Procedure 2, profiled LR, and percentile CSs should coincide for our risk-averse DGP (when $\bar{\lambda}_0$ is interior), although they may diverge for our risk-neutral DGP (when $\bar{\lambda}_0$ is on the boundary). Figure 5 compares coverage (acceptance) curves for level $\alpha = 0.95$ CSs for $\bar{\lambda}_0$ derived from all three procedures. Coverage curves are indeed nearly identical across procedures for our risk-averse DGP, further validating our approach. Meanwhile, Table 1 reports confidence bounds and medians for level $\alpha = 0.95$ CSs for $\bar{\lambda}_0$. As expected, percentile CSs exhibit strong boundary effects for our risk-neutral DGP ($\rho_0 = 0$), but coverage rates are otherwise very close to nominal levels for all methods and DGPs.

We next turn to our main Flexible copula model with $Q = 1$. In Table 2, we report coverage rates and median CS bounds for level $\alpha = 0.95$ CSs for $\bar{\lambda}_0$ derived from Procedure 2, profiled LR, and percentile CSs. We consider the AIC-preferred specification $P = 4$, as well as $P = 5$ and $P = 6$. We implement MC CSs (Procedure 2 and percentile CSs) using

[27] As explained in Appendix B, profiled LR statistics for parameters in point-identified first-price auction models can be divided into two groups: those with Gamma limiting distributions and the corresponding parameters are estimable at rate-L due to maximal order statistics $\hat{\bar{b}}_k$, $k = 1, ..., K$ for support constraints; and those with chi-square (or chi-bar) limiting distributions and the corresponding parameters, such as γ in the point-identified CRRA model ($Q = 1$), are estimable at rate-$\sqrt{L}$.

Table 1: **Frank copula,** $Q = 1$**:** Coverage rates and median bounds based on alternative CSs for $\bar{\lambda}_0 = 1/(1-\rho_0)$, parameterizing $\lambda_0(\cdot)$ as CRRA ($Q = 1$), C as Frank, and F_0 as Bernstein-P.

				Risk Averse DGP $\rho_0 = 0.5,\ \bar{\lambda}_0 = 2.0$		Risk Neutral DGP $\rho_0 = 0.0,\ \bar{\lambda}_0 = 1.0$	
Method	Prior	M	P	Coverage of $\bar{\lambda}_0$	Median CS Bounds	Coverage (see note)	Median CS Bounds
Profiled LR	—	1000	4	0.956	$[1.588, 2.392]$	0.918	$[1.000, 1.061]$
		2000	4	0.938	$[1.712, 2.277]$	0.942	$[1.000, 1.044]$
			5	0.948	$[1.705, 2.266]$	0.944	$[1.000, 1.043]$
Quantile CS	Uniform	1000	4	0.952	$[1.659, 2.388]$	0.780	$[1.002, 1.083]$
		2000	4	0.948	$[1.739, 2.265]$	0.875	$[1.001, 1.056]$
			5	0.933	$[1.734, 2.275]$	0.882	$[1.001, 1.057]$
Procedure 2	Uniform	1000	4	0.957	$[1.595, 2.383]$	0.944	$[1.000, 1.074]$
		2000	4	0.930	$[1.714, 2.271]$	0.959	$[1.000, 1.050]$
			5	0.940	$[1.703, 2.271]$	0.969	$[1.000, 1.050]$

Notes: Results are based on 500 Monte Carlo replications. MC CSs are based on $B^* = 400$ parameter draws $\{\psi^b\}_{b=1}^{400}$. For MC CSs in our risk-averse boundary DGP ($\rho_0 = 0.0$), where percentile CSs never cover $\bar{\lambda}_0$, "Coverage" reports the fraction of simulations covering $\tilde{\lambda}(x|\gamma_0)/x = 1.005$.

flat priors on all parameters except ϕ, for which we compare both flat priors and boundary priors with $\Delta_\phi = 0.01$. As expected in view of Figure 3, profiled LR coverage rates for $\bar{\lambda}_0$ are conservative for our main sample scales $M \in \{1000, 2000\}$, but approach asymptotic sizes at larger scales $M \in \{10000, 20000\}$ (which we explored for this table only as a test of our theory). Percentile CSs also demonstrate good coverage in our risk-averse DGP, but (as expected) exhibit boundary effects for our risk-neutral DGP. CSs based on Procedure 2 yield coverage rates close to nominal confidence levels in all specifications, highlighting the benefits of our robust procedure even in point-identified models. Coverage for MC CSs is not strongly sensitive to priors, but appears slightly more reliable under our $\Delta_\phi = 0.01$ boundary prior. Median CS bounds widen moving from $P = 4$ to $P = \{5, 6\}$, highlighting the finite-sample gains associated with the AIC-preferred model $P = 4$.

5.2.2 Inference for set-identified models ($Q \in \{4, 6\}$)

For our fully flexible set-identified models with $Q \in \{4, 6\}$, we compare CSs based on two numerical implementations of Procedure 2. In the first, labeled Procedure 2-KL, we approximate $PL(\psi^b)$ with $\widetilde{PL}_{KL}(\psi^b)$ obtained by slightly relaxing definition (19):

$$\widetilde{PL}_{KL}(\psi^b) = \inf\{\tilde{L}(\eta') : \eta' = \eta(\psi') \text{ for } \psi' \in \Psi \text{ with } D_{KL}(\psi^b || \psi') \leq 10^{-6}\}.$$

Table 2: **Flexible copula, Q=1:** Coverage rates and median bounds based on alternative CSs for $\bar{\lambda}_0 = 1/(1-\rho_0)$, parameterizing $\lambda_0(\cdot)$ as CRRA ($Q=1$) and $F_1^0, ..., F_K^0$ as Bernstein-P.

				Risk Averse DGP $\rho_0 = 0.5$, $\bar{\lambda}_0 = 2.0$		Risk Neutral DGP $\rho_0 = 0.0$, $\bar{\lambda}_0 = 1.0$	
Method	Prior	M	P	Coverage of $\bar{\lambda}_0$	Median CS Bounds	Coverage (see note)	Median CS Bounds
Profiled LR	—	1000	4	0.978	[1.440, 2.795]	0.988	[1.000, 1.119]
		2000	4	0.986	[1.569, 2.640]	0.990	[1.000, 1.098]
			5	0.990	[1.565, 2.820]	0.990	[1.000, 1.117]
			6	0.980	[1.493, 2.841]	0.990	[1.000, 1.138]
		10000	6	0.950	[1.708, 2.503]	0.968	[1.000, 1.063]
		20000	6	0.956	[1.737, 2.355]	0.965	[1.000, 1.051]
Quantile CS	Uniform	1000	4	0.962	[1.659, 2.659]	0.842	[1.002, 1.127]
		2000	4	0.954	[1.727, 2.508]	0.907	[1.001, 1.097]
			5	0.912	[1.767, 2.667]	0.854	[1.002, 1.129]
			6	0.946	[1.733, 2.658]	0.858	[1.002, 1.139]
	$\Delta_\phi = 0.01$	1000	4	0.962	[1.527, 2.481]	0.848	[1.002, 1.125]
		2000	4	0.966	[1.616, 2.391]	0.912	[1.001, 1.094]
			5	0.938	[1.687, 2.584]	0.827	[1.002, 1.139]
			6	0.965	[1.633, 2.530]	0.819	[1.002, 1.142]
Procedure 2	Uniform	1000	4	0.948	[1.583, 2.642]	0.967	[1.000, 1.110]
		2000	4	0.946	[1.689, 2.521]	0.960	[1.000, 1.085]
			5	0.946	[1.681, 2.647]	0.960	[1.000, 1.112]
			6	0.948	[1.657, 2.648]	0.957	[1.000, 1.120]
	$\Delta_\phi = 0.01$	1000	4	0.970	[1.532, 2.668]	0.965	[1.000, 1.107]
		2000	4	0.972	[1.629, 2.551]	0.964	[1.000, 1.083]
			5	0.954	[1.661, 2.662]	0.975	[1.000, 1.120]
			6	0.950	[1.636, 2.649]	0.961	[1.000, 1.127]

Notes: Results are based on 500 Monte Carlo replications. MC CSs are based on $B^* = 400$ parameter draws $\{\psi^b\}_{b=1}^{400}$. For MC CSs in our risk-averse boundary DGP ($\rho_0 = 0.0$), where percentile CSs never cover $\bar{\lambda}_0$, "Coverage" reports the fraction of simulations covering $\tilde{\lambda}(x|\gamma_0)/x = 1.005$.

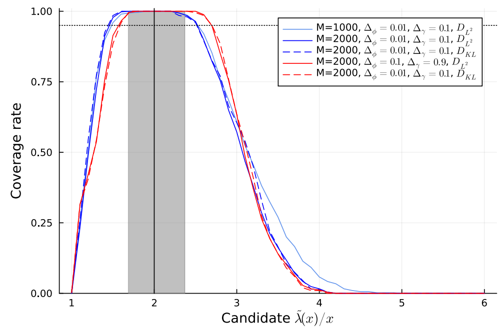
In the second, which we label Procedure 2-L^2, we approximate $PL(\psi^b)$ with $\widetilde{PL}_{L^2}(\psi^b)$ obtained by slightly relaxing the alternative definition (21) derived from Theorem 1:

$$\widetilde{PL}_{L^2}(\psi^b) = \inf\{\tilde{L}(\eta') : \eta' = \eta(\psi') \text{ for } \psi' \in \Psi \text{ with } \mathbf{s}^b = \mathbf{s}' \text{ and } D_{L^2}(\psi^b||\psi') \leq 10^{-7}\}.$$

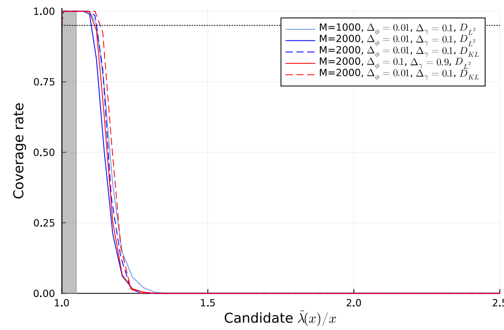
In practice, we choose tolerances 10^{-6} for $\widetilde{PL}_{KL}(\psi^b)$ and 10^{-7} for $\widetilde{PL}_{L^2}(\psi^b)$ to be small, but not so small as to undermine numerical stability.[28] By construction, both Procedure 2-KL

[28]Since $D_{KL}(\psi||\psi')$ and $D_{L^2}(\psi||\psi')$ are measured in different units, there is no reason to choose the same tolerances. We choose a smaller tolerance for $\widetilde{PL}_{L^2}(\psi^b)$ than for $\widetilde{PL}_{KL}(\psi^b)$ since evaluation of $D_{KL}(\psi^b||\psi')$ requires re-solving equilibrium bid functions in an inner loop, while evaluation of $D_{L^2}(\psi^b||\psi')$ does not.

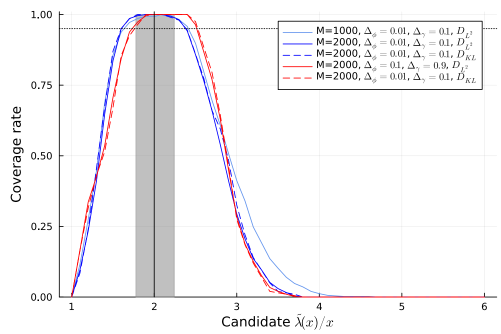
Figure 6: **Flexible copula, Q=4:** Coverage (acceptance) rates based on level $\alpha = 0.95$ CSs for $\tilde{\lambda}(x|\gamma_0)/x$ via Procedure 2 at $x = 0.2$ and $x = 0.3$, parameterizing $\lambda_0(\cdot)$ as Bernstein of order $Q = 4$ and $F_1^0, ..., F_K^0$ as Bernstein of order $P = 6$.



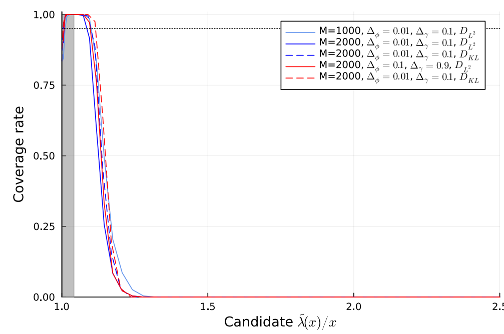
(a) $\rho_0 = 0.5$, $x = 0.2$



(b) $\rho_0 = 0.0$, $x = 0.2$



(c) $\rho_0 = 0.5$, $x = 0.3$



(d) $\rho_0 = 0.0$, $x = 0.3$

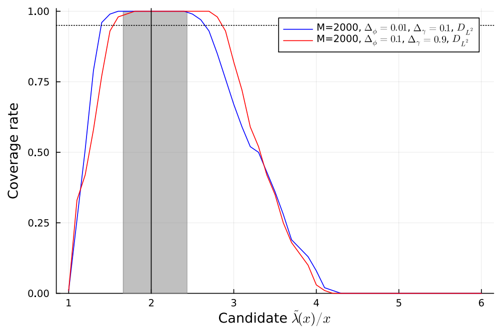
Notes: Results based on 200 replications for Procedure 2-KL and 500 for Procedure 2-L^2, with $B^* = 400$ parameter draws $\{\psi^b\}_{b=1}^{400}$ per replication. Shaded areas are profiled $D_{KL}^0(\psi) \le 10^{-6.0}$ contour sets. The vertical line is true $\tilde{\lambda}(x|\gamma_0)/x = 1/(1-\rho_0)$.

and Procedure 2-L^2 will be conservative for true identified sets, and their exact numerical results will differ. We hypothesize, however, that Procedure 2-KL will exhibit approximately correct coverage for contour sets based on $D_{KL}^0(\psi) \le 10^{-6}$.

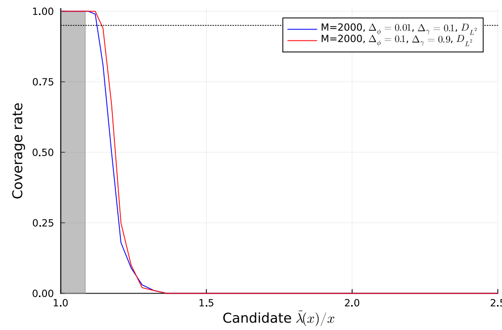
Figures 6 and 7 report estimated coverage (acceptance) curves based on level $\alpha = 0.95$ CSs for $\tilde{\lambda}(x|\gamma_0)/x$ at $x = 0.2$ and $x = 0.3$ in Flexible copula models with $Q = 4$ and $Q = 6$ respectively.[29] For $Q = 4$ and $M = 2000$, we implement both Procedure 2-KL and Procedure 2-L^2, finding that both yield very similar coverage rates. We thus focus on the simpler Procedure 2-L^2 for $Q = 6$. We also compare results based on priors with more boundary weight ($\Delta_\phi = .01$, $\Delta_\gamma = 0.1$) and less boundary weight ($\Delta_\phi = .1$, $\Delta_\gamma = 0.9$) respectively. Coverage curves vary with choice of priors, but in all specifications the sets of points covered with 95% probability correspond relatively closely to targeted $D_{KL}^0(\psi) \le 10^{-6.0}$ contour sets.

[29] We focus on $x \in \{0.2, 0.3\}$ since the maximal argument to $\tilde{\lambda}(x|\gamma_0)/x$ when $\rho_0 = 0.5$ is roughly $x \approx 0.38$.

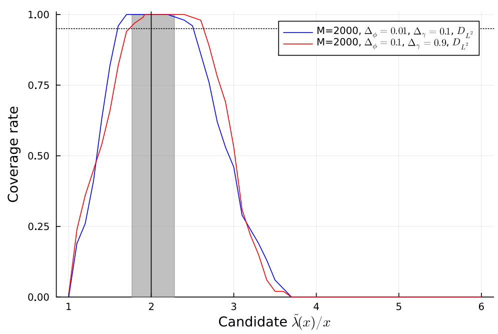
Figure 7: **Flexible copula, Q=6:** Coverage (acceptance) rates based on level $\alpha = 0.95$ CSs for $\tilde{\lambda}(x|\gamma_0)/x$ via Procedure 2 at $x = 0.2$ and $x = 0.3$, parameterizing $\lambda_0(\cdot)$ as Bernstein of order $Q = 6$ and $F_1^0, ..., F_K^0$ as Bernstein of order $P = 6$.



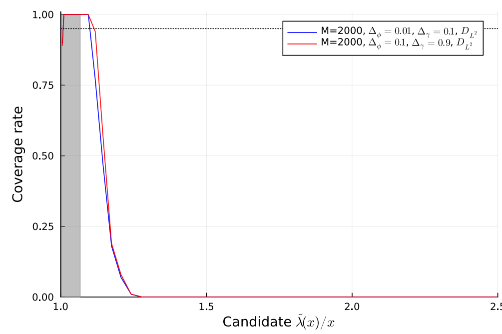
(a) $\rho_0 = 0.5$, $x = 0.2$



(b) $\rho_0 = 0.0$, $x = 0.2$



(c) $\rho_0 = 0.5$, $x = 0.3$



(d) $\rho_0 = 0.0$, $x = 0.3$

Notes: Estimates based on 100 Monte Carlo replications with $B^* = 400$ parameter draws each. Shaded areas are profiled $D_{KL}^0(\psi) \leq 10^{-6.0}$ contour sets. The vertical line is $\tilde{\lambda}(x|\gamma_0)/x = 1/(1-\rho_0)$.

Moreover, our CSs display very good power away from these sets.

We have conducted additional simulation studies, including another Monte Carlo design with a Gumbel (rather than a Frank) value-signal copula DGP, and using even more flexible Bernstein polynomial bases approximations for the utility function (up to $Q = 7$) and for latent value distributions (up to $P = 7$). Patterns from these exercises are very similar to those reported here and are not reported due to the lack of space. We view our simulation evidence as highly promising, underscoring the attractive potential of our MPEC-based implementation of CCT methods for inference in possibly set-identified auction models.

A Symmetric Monotone Equilibrium

For completeness, we first extend the simple AS-RA model presented in Section 2 to accommodate nonzero initial wealth for bidders and financial (in addition to opportunity) costs of entry. We then characterize equilibrium entry and bidding behavior within this extended model, demonstrating in the process how the more general structure considered here collapses in all economically relevant details to that in Section 2.

Potential bidders are risk averse with preferences over net post-auction wealth w described by a symmetric concave Bernoulli utility function $u(w)$. Net entry cost is given by $c(z) = c_0 + c_1(z)$, where c_0 is the financial cost of entry, $c_1(z)$ is the opportunity cost of entry, and $c_1(z)$ is strictly increasing and continuous in z. Bidders have common initial wealth $w_0 \geq c_0$.

Following LLZ, we define a normalized utility function $U(\cdot)$ as a function of the *change* in wealth x derived from bidding, normalized such that a bidder who enters the auction but does not win receives zero normalized utility:

$$U(x) \equiv u(x + w_0 - c_0) - u(w_0 - c_0).$$

For simplicity, and without loss of generality, we further normalize the scale of utility such that $U(1) = u(1 + w_0 - c_0) - u(w_0 - c_0) \equiv 1$.

As noted by LLZ, centered utility $U(\cdot)$ belongs to the same category of Arrow-Pratt absolute risk aversion (increasing, constant, or decreasing) as initial utility $u(\cdot)$. Furthermore, as we show below, knowledge of normalized utility U is equivalent to joint knowledge of non-normalized (u, w_0, c_0) with respect to characterizing equilibrium entry and bidding behavior. In this sense, the simplified presentation in the text is without loss of generality.

A.1 Equilibrium bidding

First consider the Stage 2 bidding problem faced by an entrant with valuation v_i bidding against $N-1$ potential rivals, each of whom enters when $S_j \geq \bar{s} \in (0,1)$. We seek a strictly increasing bidding strategy $\beta(\cdot|N,\bar{s})$ such that bidder i with valuation v_i optimally bids $\beta(v_i|N,\bar{s})$ when facing $N-1$ rivals who enter according to $\bar{s}$ and bid according to $\beta(\cdot|N,\bar{s})$.

Let $F(\cdot|S_j \geq \bar{s})$ denote the c.d.f. of rival j's valuation conditional on $S_j \geq \bar{s}$:

$$F(y|s_j \geq \bar{s}) = \frac{1}{1-\bar{s}} \int_{\bar{s}}^{1} F(y|t)\, dt.$$

Under Assumptions 1-4, the support of $F(\cdot|S_j \geq \bar{s})$ is a connected interval of the form $[\underline{v}(\bar{s}), \bar{v}]$, where the infimum support $\underline{v}(\bar{s})$ is differentiable in $\bar{s}$. Moreover, the density $f(\cdot|S_j \geq \bar{s})$ is locally bounded away from zero for all $v \in (\underline{v}(\bar{s}), \bar{v}]$.

Let $F^*(\cdot|N,\bar{s})$ (and $f^*(\cdot|N,\bar{s})$) be the c.d.f. (and the pdf) of the effective maximum valuation among rival entrants when i's $N-1$ rivals enter according to threshold $\bar{s}$:

$$F^*(y|N,\bar{s}) = [\bar{s} + (1-\bar{s})F(y|S_j \geq \bar{s})]^{N-1},$$

It is straightforward to show that $F^*(y|N,\bar{s})$ is increasing in $\bar{s}$ (strictly for y such that $F(y|\bar{s}) < 1$) and decreasing in N (strictly if $y < \bar{v}$).

Assuming that all potential rivals bid according to $\beta(\cdot|N,\bar{s})$, entrant i submitting bid $b_i \equiv \beta(y_i|N,\bar{s})$ will outbid all potential rivals with probability $F^*(y_i|N,\bar{s})$. The expected profit of entrant i with valuation v_i who bids *as if* her type were y_i is therefore:

$$\begin{aligned}
\pi_N(y_i, v_i; \bar{s}) &\equiv u(v_i - \beta(y_i|N,\bar{s}) + w_0 - c_0)F^*(y_i|N,\bar{s}) + u(w_0 - c_0)(1 - F^*(y_i|N,\bar{s})) \\
&= [u(v_i - \beta(y_i|N,\bar{s}) + w_0 - c_0) - u(w_0 - c_0)]F^*(y_i|N,\bar{s}) + u(w_0 - c_0) \\
&= U(v_i - \beta(y_i|N,\bar{s}))F^*(y_i|N,\bar{s}) + u(w_0 - c_0).
\end{aligned}$$

Taking a first-order condition of the final expression with respect to y_i, enforcing the equilibrium condition $y_i = v_i$, and solving for $\beta'(\cdot|N,\bar{s})$, we conclude that $\beta(\cdot|N,\bar{s})$ must satisfy

$$\beta'(v_i|N,\bar{s}) = \frac{U(v_i - \beta(v_i|N,\bar{s}))}{U'(v_i - \beta(v_i|N,\bar{s}))} \frac{f^*(v_i|N,\bar{s})}{F^*(v|N,\bar{s})}, \tag{24}$$

subject to the boundary condition $\beta(\underline{v}(\bar{s})|N,\bar{s}) = p_0$. LLZ show that (24) yields a unique solution $\beta(\cdot|N,\bar{s})$ which is strictly increasing and differentiable in v, strictly increasing in N, and strictly decreasing and continuous in $\bar{s}$ for $\bar{s} \in (0,1]$. Since $F^*(y|N,\bar{s})$ is increasing in $\bar{s}$ and decreasing in N, bidder i's expected equilibrium Stage 2 profit

$$\pi^*_N(v_i; \bar{s}) \equiv U(v_i - \beta(v_i|N,\bar{s}))F^*(v_i|N,\bar{s}) + u(w_0 - c_0)$$

will therefore be strictly increasing and continuous in v_i, strictly decreasing in N, and increasing (strictly for $v_i > \underline{v}(\bar{s})$) and continuous in $\bar{s}$ for $\bar{s} \in (0,1]$.

It remains to consider bidding when $\bar{s} = 0$. In this case, the bidder with the lowest valuation can no longer win with positive probability by submitting a bid equal to the reserve price. The boundary condition characterizing equilibrium bidding therefore changes discontinuously from $\beta(\underline{v}(\bar{s})|N,\bar{s}) = p_0$ for $\bar{s} > 0$ to $\beta(\underline{v}|N,0) = \underline{v}$ for $\bar{s} = 0$. $\beta(\cdot|N,0)$ is still given by the unique solution to the initial value problem (24), with the boundary condition $\beta(\underline{v}|N,0) = \underline{v}$ replacing the boundary condition $\beta(\underline{v}(\bar{s})|N,\bar{s}) = p_0$ above.

A.2 Equilibrium entry

Now consider the Stage 1 entry decision of potential bidder i with signal s_i facing $N-1$ potential rivals who enter according to $\bar{s}$ and bid according to $\beta(\cdot|N,\bar{s})$. Recall that i must forego opportunity costs $c_1(z)$ from staying out. Holding the opportunity cost shifter

z constant, the *change* in payoff i expects from entry is therefore:

$$\int_{\underline{v}}^{\bar{v}} \pi_N^*(v;\bar{s})\, dF(v|S_i = s_i) - u(w_0 + c_1(z))$$
$$= \int_{\underline{v}}^{\bar{v}} U(v - \beta(v|N,\bar{s}))F^*(v|N,\bar{s})\, dF(v|S_i = s_i) + u(w_0 - c_0) - u(w_0 + c_1(z)).$$

Noting that $u(w_0 + c_1(z)) = u(c_0 + c_1(z) + w_0 - c_0)$, we may rewrite the final line as

$$\pi^e(s_i, \bar{s}, N) - U(c(z)),$$

where $\pi^e(s_i, \bar{s}, N)$ denotes the expected *normalized* post-entry profit of a bidder with signal $S_i = s_i$, facing $N-1$ potential rivals who enter according to threshold $\bar{s}$:

$$\pi^e(s_i, \bar{s}, N) \equiv \int_{\underline{v}}^{\bar{v}} U(v - \beta(v|N,\bar{s}))F^*(v|N,\bar{s})\, dF(v|S_i = s_i).$$

Finally, consider the threshold $s_N(z)$ characterizing *equilibrium* entry at (N, z). If $s_N(z) \in (0,1)$, then a potential bidder with signal $s_i = s_N(z)$ must be indifferent to entry, i.e.:

$$\pi^e(s_N(z), s_N(z), N) \equiv U(c(z)). \tag{25}$$

For $\bar{s} > 0$, the properties above imply that $\pi^e(s_i, \bar{s}, N)$ is increasing and continuous in s_i, strictly increasing and continuous in $\bar{s}$, and decreasing in N. If, in addition, $\pi^e(\bar{s}, \bar{s}, N)$ is continuous in $\bar{s}$ as $\bar{s} \to 0^+$—a property we establish in the proof of Theorem 4 below—then there will exist a unique, continuous function $s_N(z)$ satisfying the equilibrium conditions above. Taken together, these properties together imply the following result:

Theorem 4. *Suppose that $U \in \mathcal{U}$, $F \in \mathcal{F}$, and $C \in \mathcal{C}$. Then there exists a unique symmetric monotone pure strategy Bayesian Nash Equilibrium for any $N \in \mathcal{N}$ and $z \in \mathcal{Z}$. The equilibrium entry threshold $s_N(z)$ is uniquely determined as follows: if $\pi^e(0,0,N) > U(c(z))$, then $s_N(z) = 0$ and all bidders enter; if $\pi^e(1,1,N) < U(c(z))$, then $s_N(z) = 1$ and no bidder enters; otherwise, $s_N(z)$ is the unique solution to $\pi^e(s_N(z), s_N(z), N) = U(c(z))$. Moreover, $s_N(z)$ is increasing in both N and $c(z)$, strictly if $s_N(z) \in (0,1)$, and if $s_N(z) < 1$ for any N, then $s_N(z) < 1$ for all N.*

The equilibrium bidding strategy $\beta(\cdot|N, s_N(z))$ is the unique solution to the initial value problem (24) with $\bar{s} = s_N(z)$, subject to the boundary condition $\beta(\underline{v}(s_N(z))|N, s_N(z)) = p_0$ if $s_N(z) > 0$, or $\beta(\underline{v}|N, s_N(z)) = \underline{v}$ if $s_N(z) = 0$.

B Profile LR subvector inference in set-identified models with parameter-dependent support

In this Appendix, we present a new Theorem 5 on the frequentist validity of CCT's Procedure 2 for set identified subvectors in models with parameter dependent support. We also discuss some sufficient conditions for frequentist validity of CCT's CSs, and verify these analytically in a simple set identified first-price auction example.

B.1: Subvector inference

Consider a parametric model $\mathcal{P} = \{p_\psi : \psi \in \Psi\}$ where $p_\psi(\cdot)$ is a probability density with respect to a common dominating measure λ. Let $D_{KL}(p\|q) = \int p \log(p/q) d\lambda$ denote the Kullback-Leibler divergence, and $p^o \in \mathcal{P}$ be the true density of the data X. A natural population criterion is $\mathbb{L}(\psi) = \mathbb{E}[\log p_\psi(X)] = \mathbb{E}[\log p^o(X)] - D_{KL}(p^o\|p_\psi)$, where $\mathbb{E}[\cdot]$ denotes the expectation with respect to p^o. We assume that $\mathbb{L}(\psi)$ is an upper semicontinuous function of ψ with $\sup_{\psi\in\Psi} \mathbb{L}(\psi) < \infty$. The *identified set* for $\psi \equiv (\mu, \delta) \in \Psi$ is:

$$\Psi_I := \{\psi \in \Psi : D_{KL}(p^o\|p_\psi) = 0\} = \left\{\psi \in \Psi : \mathbb{L}(\psi) = \sup_{\varphi\in\Psi} \mathbb{L}(\varphi)\right\} . \tag{26}$$

Let $\mathbf{X}_n = (X_1, \ldots, X_n)$ denote a random sample of size n from the joint density $\prod_{i=1}^n p^o(X_i)$ (recall that L is used to denote sample size in the main text). Let $\mathcal{L}(\psi) = \sum_{i=1}^n \log p_\psi(X_i)$ and $\hat\psi \in \Psi$ be an approximate MLE, i.e., $\mathcal{L}(\hat\psi) = \sup_{\psi\in\Psi} \mathcal{L}(\psi) + o_p(1)$. The log-likelihood ratio (LR) for $\psi = (\mu, \delta) \in \Psi$ is $\mathcal{Q}(\psi) \equiv 2[\mathcal{L}(\hat\psi) - \mathcal{L}(\psi)]$.

Let $\mathcal{M} = \{\mu : (\mu, \delta) \in \Psi \text{ for some } \delta\}$ and $H_\mu = \{\delta : (\mu, \delta) \in \Psi\}$. The identified set for the subvector $\mu \in \mathcal{M}$ is $\mathcal{M}_I := \{\mu : (\mu, \delta) \in \Psi_I \text{ for some } \delta \in H_\mu\}$, or equivalently

$$\mathcal{M}_I = \{\mu : \inf_{\delta\in H_\mu} D_{KL}(p^o\|p_{(\mu,\delta)}) = 0\} = \{\mu : \sup_{\delta\in H_\mu} \mathbb{L}(\mu, \delta) = \sup_{\varphi\in\Psi} \mathbb{L}(\varphi)\}. \tag{27}$$

For any given $\psi \in \Psi$, we define a collection of $\mu \in \mathcal{M}$ as follows:

$$\mathcal{M}(\psi) := \{\mu \in \mathcal{M} : D_{KL}(p_\psi\|p_{(\mu,\delta)}) = 0 \text{ for some } \delta \in H_\mu\}.$$

Define the (sample) profile LR for the set $\mathcal{M}(\psi)$ as

$$PQ_n(\mathcal{M}(\psi)) \equiv \sup_{\mu\in\mathcal{M}(\psi)} \inf_{\delta\in H_\mu} \mathcal{Q}(\mu, \delta) . \tag{28}$$

Given $\mathcal{L}$ and a prior Π over Ψ, the posterior distribution Π_n for ψ given $\mathbf{X}_n$ is

$$d\Pi_n(\psi|\mathbf{X}_n) = \frac{\exp[\mathcal{L}(\psi)]d\Pi(\psi)}{\int_\Psi \exp[\mathcal{L}(\psi)]d\Pi(\psi)} = \frac{\exp[-0.5\mathcal{Q}(\psi)]d\Pi(\psi)}{\int_\Psi \exp[-0.5\mathcal{Q}(\psi)]d\Pi(\psi)} . \tag{29}$$

We recall CCT's Procedures 1 and 2 in terms of LR statistics as follows:

- Draw a sample $\{\psi^1, \ldots, \psi^{B^*}\}$ from the posterior distribution Π_n in (29).
- Procedure 1. Let $\xi^{mc}_{n,\alpha}$ be the α quantile of $\left\{\mathcal{Q}(\psi^b)\right\}_{b=1}^{B^*}$. Compute the $100\alpha\%$ confidence set for Ψ_I as

$$\hat\Psi_\alpha = \{\psi \in \Psi : \mathcal{Q}(\psi) \le \xi^{mc}_{n,\alpha}\} . \tag{30}$$

- Procedure 2. Let $\xi^{mc,p}_{n,\alpha}$ be the α quantile of $\left\{PQ_n(\mathcal{M}(\psi^b))\right\}_{b=1}^{B^*}$. Compute the 100$\alpha$% confidence set for $\mathcal{M}_I$ as

$$\hat{M}_\alpha = \left\{\mu \in \mathcal{M} : \inf_{\delta \in H_\mu} \mathcal{Q}(\mu, \delta) \leq \xi^{mc,p}_{n,\alpha}\right\}. \tag{31}$$

The following is a list of assumptions from CCT that allows for set identified models with parameter dependent support. See Appendix B.2 below for discussions.

Assumption 5. (CCT Assumption 4.1, posterior contraction)
(i) $\mathcal{L}(\hat{\psi}) = \sup_{\psi \in \Psi_{osn}} \mathcal{L}(\psi)$, with $(\Psi_{osn})_{n\in\mathbb{N}}$ a sequence of local neighborhoods of Ψ_I;
(ii) $\Pi_n(\Psi^c_{osn} | \mathbf{X}_n) = o_p(1)$, where $\Psi^c_{osn} = \Psi \backslash \Psi_{osn}$.

We presume the existence of a fixed neighborhood Ψ^N_I of Ψ_I (with $(\Psi_{osn})_{n\in\mathbb{N}} \subset \Psi^N_I$ for all n sufficiently large) upon which there is a local reduced-form reparameterization $\psi \mapsto \varpi(\psi)$ from Ψ^N_I into $\Upsilon \subseteq \mathbb{R}^{d^*}$ for some $d^* \in [1, \infty)$, such that $\varpi(\psi) = 0$ if and only if $\psi \in \Psi_I$. Let $\hat{\varpi} \equiv \varpi(\hat{\psi})$.

Assumption 6. (CCT Assumption C.2, local fan-shaped LR contour)
(i) There is a mapping $h : \Upsilon \mapsto \mathbb{R}_+$ and a sequence $\{a_n > 0\}_{n\in\mathbb{N}}$ with $a_n \to 0$ such that:

$$\sup_{\psi \in \Psi_{osn}} \left| \frac{\frac{a_n}{2}\mathcal{Q}(\psi) - h(\varpi(\psi) - \hat{\varpi})}{h(\varpi(\psi) - \hat{\varpi})} \right| = o_p(1)$$

with $\sup_{\psi \in \Psi_{osn}} \|\varpi(\psi)\| \to 0$ and $\inf\{h(\varpi) : \|\varpi\| = 1\} > 0$;
(ii) there exist $r_1, \ldots, r_{d^*} > 0$ such that $th(\varpi) = h(t^{r_1}\varpi_1, t^{r_2}\varpi_2, \ldots, t^{r_{d^*}}\varpi_{d^*})$ for each $t > 0$;
(iii) the sets $K_{osn} = \{(c_n^{-r_1}(\varpi_1(\psi) - \hat{\varpi}_1), \ldots, c_n^{-r_{d^*}}(\varpi_{d^*}(\psi) - \hat{\varpi}_{d^*}))' : \psi \in \Psi_{osn}\}$ cover $\mathbb{R}^{d^*}_+$ for any positive sequence $(c_n)_{n\in\mathbb{N}}$ with $c_n \to 0$ and $a_n/c_n \to 1$.

Let $r^* \equiv \sum_{i=1}^{d^*} r_i$. Let $\mathcal{G}_{r^*} := \text{Gamma}(r^*, 2)$ denote a Gamma random variable that is Gamma distributed with shape parameter r^* and scale parameter 2. By slightly modifying the arguments in Fan, Hung, and Wong (2000) one can show that, under Assumption 6,

$$\sup_{\psi \in \Psi_I} \mathcal{Q}(\psi) = 2a_n^{-1}h(-\hat{\varpi}) \times [1 + o_p(1)] \rightsquigarrow \mathcal{G}_{r^*} \;.$$

Remark 1. *Assumption 6 is typically satisfied with $a_n = c_n = n^{-1}$, $d^* = d_1^* + d_2^*$, $\varpi(\psi)' = (\boldsymbol{r}_1(\psi)', \boldsymbol{r}_2(\psi)')$ with $\boldsymbol{r}_j \in \mathbb{R}^{d_j^*}$ for $j = 1, 2$, and $\varpi(\psi) = 0$ iff $\psi \in \Psi_I$. The MLE $\hat{\varpi} = (\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_2)$ is such that $\sqrt{n}\hat{\boldsymbol{r}}_1 = O_p(1)$ is asymptotically d_1^*-dimensional normal (provided $d_1^* > 0$), $n\hat{\boldsymbol{r}}_2 = O_p(1)$ is asymptotically d_2^*-dimensional exponential (provided $d_2^* > 0$) and independent of $\sqrt{n}\hat{\boldsymbol{r}}_1$. There are two non-negative mappings h_1, h_2 such that $2nh(\varpi(\psi) - \hat{\varpi}) = 2h_1(\sqrt{n}[\boldsymbol{r}_1 - \hat{\boldsymbol{r}}_1]) + 2h_2(n[\boldsymbol{r}_2 - \hat{\boldsymbol{r}}_2])$, and $2h_1(-\sqrt{n}\hat{\boldsymbol{r}}_1) \rightsquigarrow \mathcal{G}_{0.5d_1^*}$ and $2h_2(-n\hat{\boldsymbol{r}}_2]) \rightsquigarrow \mathcal{G}_{d_2^*}$. Hence $\sup_{\psi \in \Psi_I} \mathcal{Q}(\psi) \rightsquigarrow \mathcal{G}_{r^*}$ with $r^* = 0.5d_1^* + d_2^*$.*

Let Π_Υ be the image measure (under the map $\psi \mapsto \varpi(\psi)$) of the prior Π on Ψ_I^N, namely $\Pi_\Upsilon(A) = \Pi(\{\psi \in \Psi_I^N : \varpi(\psi) \in A\})$. Let $B_\epsilon \subset \mathbb{R}^{d^*}$ be a closed ball of radius ϵ centered at the origin.

Assumption 7. (CCT Assumption 4.3, prior)
(i) $\int_\Psi e^{\mathcal{L}(\psi)} d\Pi(\psi) < \infty$ almost surely;
(ii) Π_Υ has a continuous, strictly positive density π_Υ on $B_\epsilon \cap \Upsilon$ for some $\epsilon > 0$.

Let $\xi^{post}_{n,\alpha}$ denote the α-th quantile of $\mathcal{Q}(\psi)$ under the posterior distribution Π_n.

Assumption 8. (CCT Assumption 4.4, MC convergence) $\xi^{mc}_{n,\alpha} = \xi^{post}_{n,\alpha} + o_p(1)$.

The next lemma collects CCT's Lemma C.1 and Theorem C.1 for easy reference.

Lemma 1. *(CCT's Lemma C.1 and Theorem C.1) Let Assumptions 5, 6 and 7 hold. Then:*
(1) $\sup_y |\Pi_n(\{\psi : \mathcal{Q}(\psi) \le y\}|\mathbf{X}_n) - Pr(\mathcal{G}_{r^*} \le y)| = o_p(1)$.
(2) In addition if Assumption 8 holds, then $\lim_{n\to\infty} Pr(\Psi_I \subseteq \hat{\Psi}_\alpha) = \alpha$.

The next assumption replaces CCT's Assumption 4.5 for profile LR by a more general fan-shaped likelihood contour set.

Assumption 9. There exists a quasi-convex function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that:

$$\sup_{\psi \in \Psi_{osn}} \left| PQ_n(\mathcal{M}(\psi)) - f\left(2a_n^{-1} h(\varpi(\psi) - \hat{\varpi}_n)\right)\right| = o_p(1)$$

Assumption 9 implies that the profile LR for the set $\mathcal{M}_I$ (for μ) satisfies

$$PQ_n(\mathcal{M}_I) \equiv \sup_{\mu \in \mathcal{M}_I} \inf_{\delta \in H_\mu} \mathcal{Q}(\mu, \delta) \rightsquigarrow f(\mathcal{G}_{r^*})\,. \tag{32}$$

Let $\xi^{post,p}_{n,\alpha}$ denote the α quantile of the profile LR $PQ_n(\mathcal{M}(\psi))$ under the posterior distribution Π_n.

Assumption 10. (CCT Assumption 4.6, MC convergence) $\xi^{mc,p}_{n,\alpha} = \xi^{post,p}_{n,\alpha} + o_p(1)$.

Theorem 5. *Let Assumptions 5, 6, 7, 9 and 10 hold. Then:*

$$\lim_{n\to\infty} Pr(\mathcal{M}_I \subseteq \hat{M}_\alpha) \ge \alpha\,.$$

B.2: Discussion and an auction example

Most assumptions above, including Assumptions 5 and 7, have been discussed in detail in CCT. For point-identified models $\Psi_I = \{\psi_0\}$, Assumption 6 or Remark 1 (for $\varpi(\psi) = \psi - \psi_0$ or a one-to-one transformation) assumes that the log-likelihood ratio has a "local fan-shaped contour" in the sense of Fan, Hung, and Wong (2000), which is a substantially weaker requirement than the classical condition of a local elliptical contour (i.e., quadratic expansion), and accommodates, among other things, parameter dependent support. In particular, Remark 1 is satisfied by many parametric point identified first-price auction models including Donald and Paarsch (1993), Hirano and Porter (2003) and Chernozhukov and Hong (2004). For partially-identified models, as a proof device, CCT presume existence of a point-identified local reduced-form parameter vector $\varpi(\psi)$ for which the "local fan-shaped contour" condition (Assumption 6) holds.[30] Neither d^*, the effective dimension of the reduced-form parameters, nor r^* need to be known in the implementation of Procedures 1 and 2 CSs.

In complicated auction models such as ours, the point-identified reduced-form parameters $\varpi(\psi) = (\mathbf{r}_1(\psi), \mathbf{r}_2(\psi))$ in Remark 1 (or Assumption 6) are the equilibrium bid distributions and entry thresholds at each competition level (or one-to-one transformations thereof). For the models in Section 5, we expect that Remark 1 is satisfied by letting $\mathbf{r}_1(\psi) \in \mathbb{R}^{d_1^*}$ relate to the global shape of the predicted equilibrium bid density $g(b_i|\psi)$, and $\mathbf{r}_2(\psi) \in \mathbb{R}^{d_2^*}$ relate to the support constraints $\beta_k(\bar{v}|\psi) - \beta_k(\bar{v}_0|\psi_0) = 0,\ k = 1, ..., K$. The MLE $\hat{\varpi} = \varpi(\hat{\psi}) = (\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = \arg\max\{\mathcal{L}(\psi) : \beta_k(\bar{v}|\psi) \geq \hat{\bar{b}}_k,\ k = 1, ..., K\}$. Unfortunately, for both models presented in Section 5, the equilibrium bid function $\beta_k(\cdot|\psi)$, the bid density $g(b_i|\psi)$, and hence the log-likelihood functions $\mathcal{L}(\psi)$ do not have analytic expressions, which renders analytic verification of Remark 1 (or Assumption 6) and CCT's regularity conditions challenging. Crucially, however, the key prediction that posterior and frequentist LR statistics follow Gamma$(r^*, 2)$ distributions can be validated through simulation, as in Figure 4 of Section 5. This is sufficient for CSs based on CCT's Procedures 1 and 2 to have asymptotically valid frequentist coverage for the relevant identified sets.

In some very simple first-price auction models in which the functions $\beta_k(\cdot|\psi)$, $g(b_i|\psi)$ and $\mathcal{L}(\psi)$ have closed-form solutions, Remark 1 (or Assumption 6) can be verified analytically. For example, suppose that bidders have CRRA utility $U(x) = x^{1-\rho}$ for $\rho \in [0, 1]$, with valuations drawn i.i.d. from a power distribution $F(v) = (v/\bar{v})^{\phi}$ on $[0, \bar{v}]$ with shape parameter $\phi > 0$. The auction model parameters are $\psi = (\phi, \gamma, \bar{v})$ with $\gamma = 1 - \rho \in [0, 1]$. We observe an i.i.d. sample of bids generated with true parameters $\psi_0 = (\phi_0, \gamma_0, \bar{v}_0)$ at one competition

[30] Importantly, whereas CCT's Assumption 4.2 (local quadratic expansion), used in their main text, fails to hold with parameter dependent support even in point-identified models, Assumption 6 allows for both partially-identified models and parameter dependent support.

level $K = 1$, with $N \geq 2$ bidders who enter with certainty. Let $N^* = N - 1$. For a given ψ, the equilibrium bid function is $\beta(v|\psi) = [\phi N^*/(\gamma + \phi N^*)]v$, the predicted maximum bid is $\bar{b}(\psi) \equiv \beta(\bar{v}|\psi)$ with $\bar{b}^0 \equiv \beta(\bar{v}_0|\psi_0)$ as the population true maximum bid. The predicted bid density is $g(b_i|\psi) = \phi[\bar{b}(\psi)]^{-\phi}(b_i)^{\phi-1} \times 1\{0 \leq b_i \leq \bar{b}(\psi)\}$. The true bid density is $p^o(b_i) = g(b_i|\psi_0)$. The true parameter $\psi_0 = (\phi_0, \gamma_0, \bar{v}_0)$ is partially identified with the identified set

$$\Psi_I = \left\{(\phi, \gamma, \bar{v}) \in (0, A) \times [0, 1] \times [0, V] : \phi = \phi_0 = (\log(\bar{b}^0) - \mathbb{E}[\log(b_i)])^{-1},\ \bar{b}(\phi, \gamma, \bar{v}) = \bar{b}^0\right\}.$$

The reduce-form parameter is $\varpi(\psi) = (\phi - \phi_0, \bar{b}(\psi) - \bar{b}^0)$ with its estimate $\hat{\varpi} = (\hat{\phi} - \phi_0, \hat{\bar{b}} - \bar{b}^0)$, where $\hat{\bar{b}} = \max_{1 \leq i \leq n} b_i$ is the MLE of $\bar{b}^0$ and $\hat{\phi}$ is the MLE of ϕ_0:

$$\hat{\phi} = \arg\max_{\phi} \left\{ \max_{(\gamma,\bar{v}):\bar{b}(\phi,\gamma,\bar{v}) \geq \hat{\bar{b}}} \sum_{i=1}^{n} \log(g(b_i|\psi)) \right\} = \left[\log(\hat{\bar{b}}) - \frac{1}{n}\sum_{i=1}^{n} \log(b_i)\right]^{-1}.$$

It is easy to check that $\sqrt{n}(\phi_0)^{-1}(\hat{\phi} - \phi_0) \rightsquigarrow N(0, 1)$ and that $n(\bar{b}^0 - \hat{\bar{b}})$ is asymptotically shifted exponential with mean $(\bar{b}^0/\phi_0)$ as its mean and standard derivation. Also

$$0.5\mathcal{Q}(\psi) = n\log(\hat{\phi}/\phi) + n\phi/\hat{\phi} - n + n\phi\log(\bar{b}(\psi)/\hat{\bar{b}}) \times 1\{\bar{b}(\psi) \geq \hat{\bar{b}}\},$$

$$\sup_{\psi \in \Psi_I} \mathcal{Q}(\psi) = n\left(\frac{\hat{\phi} - \phi_0}{\phi_0}\right)^2 \times [1 + o_p(1)] + 2n\phi_0 \log(\bar{b}^0/\hat{\bar{b}}) \times 1\{\bar{b}^0 \geq \hat{\bar{b}}\} \rightsquigarrow \mathcal{G}_{r^*}, \quad r^* = 1.5\ .$$

Remark 1 (or Assumption 6) is satisfied with $a_n = c_n = n^{-1}$. $d^* = 2$, $r_1 = 0.5$, $r_2 = 1$, $r^* = 1.5$, and for all $\psi \in \Psi_{osn}$,

$$2nh(\varpi(\psi) - \hat{\varpi}) = \left(\sqrt{n}[\varpi_1(\psi) - \hat{\varpi}_1]/\phi_0\right)^2 + 2\phi_0 \frac{n[\varpi_2(\psi) - \hat{\varpi}_2]}{\bar{b}^0} \times 1\{[\varpi_2(\psi) - \hat{\varpi}_2] \geq 0\}\ .$$

The identified set for γ is $\mathcal{M}_I = \left[0,\ \min\{((\bar{b}^0)^{-1}V - 1)N^*\phi_0, 1\}\right]$ provided that $V \geq \bar{b}^0_1$. Let $\bar{\gamma} := ((\bar{b}^0)^{-1}V - 1)N^*\phi_0 < 1$ and $\hat{\bar{\gamma}} = ((\hat{\bar{b}})^{-1}V - 1)N^*\hat{\phi}$. Then $\sqrt{n}[(\bar{\gamma}/\hat{\bar{\gamma}}) - 1] = (1 + o(1))\sqrt{n}[(\phi_0/\hat{\phi}) - 1]$. We can show that $PQ_n(\mathcal{M}_I) \rightsquigarrow W \leq \chi^2_1$, where χ^2_1 is a random variable whose distribution is standard chi squared one. Thus CCT's Procedure 3 CS is also valid.

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