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Stability in Large Markets

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In matching models, pairwise stable outcomes do not generally exist without substantial restrictions on both preferences and the topology of the network of contracts. We address the foundations of matching markets by developing a matching model with a continuum of agents that allows for arbitrary preferences and network structures. We show that pairwise stable outcomes are guaranteed to exist. When agents can interact with multiple other counterparties, pairwise stability is too weak of a solution concept, and we argue that a refinement of it called *tree stability* is the most appropriate solution concept in this setting. Our main results show tree-stable outcomes exist for arbitrary preferences and network topologies.

Key words: matching, large markets, complementarities, trading networks, tree stability.

1. INTRODUCTION

Matching models based on Gale and Shapley (1962) are well-suited to analyzing highly differentiated markets in which agents interact via personalized contracts (Crawford and Knoer, 1981; Kelso and Crawford, 1982; Roth, 1984; Ostrovsky, 2008). These models typically employ equilibrium concepts based on *pairwise stability* (Gale and Shapley, 1962), which requires that there be no mutually desirable unsigned "blocking" contract between any pair of agents.

Matching frameworks can also naturally capture interconnected markets, such as production networks in which the technologies of successive intermediaries transform initial inputs into final goods (Ostrovsky, 2008; Hatfield et al., 2013). Key features of production networks include pervasive complementarities among inputs and technologies (see, e.g., Milgrom and Roberts (1990)), as well as complex network topologies (see, e.g., Dhyne et al. (2021)). However, pairwise stable outcomes do not generally exist in the presence of complementarities (see, e.g., Kelso and Crawford (1982) and Roth (1984)), or for markets that lack a two-sided structure (Gale and Shapley, 1962). This nonexistence is a fundamental problem for applying matching models to interconnected markets: incorporating realistic features into preferences and the structure of the market can make the models lack predictive power.

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In this paper, we address the foundations of matching markets with complex preferences by focusing on large markets. Formally, we consider a market with a continuum of agents—following Aumann's (1964, 1966) work on general equilibrium theory and Azevedo and Hatfield's (2018) work on matching markets. We show that pairwise stable outcomes are guaranteed to exist without restrictions on preferences. In particular, this existence result applies in the presence of arbitrary forms of complementarities. However, when agents can participate in multiple interactions, pairwise stability is too weak of a solution concept. We therefore focus on Ostrovsky's (2008) concept of *tree stability*, a refinement of pairwise stability that allows agents to form blocks consisting of acyclic networks of contracts. Our main results show that tree-stable outcomes are guaranteed to exist without conditions on preferences or the network structure—unlike for stronger solution concepts.

To understand how working with a continuum of agents addresses the existence issues from the case of finite markets, let us revisit Gale and Shapley's (1962) classic "roommates problem." Suppose that Alex, Blake, and Cameron are considering sharing a two-person apartment. Each would prefer to have a roommate rather than living alone. Alex would prefer to share with Blake, Blake would prefer to share with Cameron, and Cameron would prefer to share with Alex. In a finite market, there is no pairwise stable matching. For example, if Alex and Blake were roommates and Cameron lived alone, then Blake and Cameron would comprise a block: they would both rather live together instead. But the situation is different with a continuum of agents. For example, with unit-mass continuums of agents of types Alex, Blake, and Cameron, there are pairwise stable matchings in which $\frac{1}{2}$ mass of agents of each type are roommates with agents of each other type.[1]

Our results build on the insights of the preceding example. In our model, there is a continuum of agents that is divided into finitely many types (see, e.g., Scarf (1962) and Azevedo and Hatfield (2018)). The types are the nodes of a network, whose edges represent the contracts that agents can sign. The network structure can capture production linkages in the economy (Ostrovsky, 2008; Hatfield et al., 2013). We allow for a general network topology—including two-sided markets, as well as vertical supply chains (Ostrovsky, 2008) and settings with horizontal trade between intermediaries (Hatfield et al., 2013; Fleiner et al., 2018; Fleiner, Jagadeesan, Jankó, and Teytelboym, 2019).

We show that pairwise-stable outcomes are guaranteed to exist in large matching markets. This existence result applies for arbitrary preferences, including in the presence of arbitrary complementarities between contracts. It also applies for arbitrary network topologies, but is a new result even in the case of two-sided markets.

When agents can sign multiple contracts, however, it is natural that they be able to coordinate on more complex blocks than ones that consist of single contracts. The main part of our analysis refines the existence result for pairwise stability to consider deviations consisting of acyclic networks of contracts. The corresponding stability concept is *tree stability*, which was introduced for finite markets by Ostrovsky (2008). For intuition, suppose that agents find blocks via an iterative process under which agents can offer multiple contracts at once, and agents who receive offers can propose further new contracts in turn (Vocke, 2023). Then, in large markets, an agent making an offer almost

1. In a setting without a continuum of agents, Aharoni and Fleiner (2003) constructured an analogous matching and interpreted it as "fractional stable matching" by exploiting the unit-demand structure of the roommates problem. See also Tan (1991).

never leads to the same agent later receiving an offer—so only acyclic deviations can be found.

Our main result shows that tree-stable outcomes are guaranteed to exist in large markets for arbitrary preferences and network topologies. This existence result starkly contrasts with the case of finite markets discussed above. It also contrasts with previous results on large markets. Indeed, in large markets, Azevedo and Hatfield (2018) considered *stability*—a stronger solution concept than tree stability that allows for deviations that include cycles.[2] They showed that stable outcomes do not generally exist in the presence of complementarities (even in two-sided markets), or in networks other than vertical supply chains. Given the pervasiveness of complementarities in (and the complexity of topologies of) real-world production networks, these negative results are problematic for stability as a solution concept. By contrast, our results show that pairwise stability and tree stability are compatible with all preferences and network structures in large markets.

A key implicit assumption underlying large-market models with a continuum of agents is that agents within a continuum are "anonymous" (Aumann, 1964) or "symmetric" (Milnor and Shapley, 1978). However, in practice, interactions between small groups of agents are typically non-anonymous, especially in matching markets (e.g., firms form long-lasting relationships with groups of counterparties). If a group of agents can interact non-anonymously, then they may be able to make contracts among themselves contingent on one another, and therefore improve coordination.

To capture the coordination possibilities present in small markets while maintaining the tractability offered by models with a continuum of agents, we introduce an extension with contingent contracts. In the extended model, we allow some—but generally not all—subnetworks of contracts to be bundled together into single interactions (Hatfield and Kominers, 2017; Rostek and Yoder, 2020), and therefore be made contingent on each other. These contingencies limit deviations by forcing agents to drop all contracts involved in a bundle if they desire to drop any of the involved contracts, but also allow for new deviations by increasing agents' ability to commit to complex deviations. In particular, bundles whose underlying contract network involves cycles can be part of acyclic blocks in our extension. Thus, our extended model allows for the possibility that groups that interact non-anonymously can also coordinate on complex deviations.

Previous work has focused on the extreme case in which no subnetworks can be bundled, sharply limiting agents' coordination power. Our other main results establish that in the other extreme case in which all subnetworks of contracts can be bundled, tree stability essentially coincides with stability and the core. Thus, incorporating bundling allows us to not only cover the extreme cases with limited and full coordination power considered respectively in matching and cooperative game theory, but also to smoothly interpolate across cases with intermediate coordination power within a unified framework. Our existence results also carry over to the extension with bundling—showing that tree-stable outcomes exist for arbitrary degrees of coordination power.

Related literature. In the matching literature, most previous analyses have taken a different approach to ensuring existence: they have ruled out complementarities (or placed severe restrictions on their form), and constrained the topology of the

2. Hatfield and Milgrom (2005) and Hatfield and Kominers (2012, 2017) introduced this notion of stability.

network of contracts. For example, Kelso and Crawford (1982), Roth (1984), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), and Hatfield and Kominers (2017) considered two-sided matching markets and ruled out complementarities. On the other hand, Ostrovsky (2008), Westkamp (2010), Hatfield and Kominers (2012), Hatfield et al. (2013), Fleiner et al. (2018), and Fleiner, Jagadeesan, Jankó, and Teytelboym (2019) modeled matching in networks, but required that contracts have natural “buyer” and “seller” counterparties and placed restrictions on preferences that rule out, for example, complementarities between inputs in production.[3]

A recent strand of the matching literature has developed existence results for two-sided matching models with a continuum of agents that allow for some more general preferences. In this strand, Che et al. (2019) showed the existence of stable outcomes in large-market many-to-one settings with complementarities.[4] Unlike Che et al. (2019), we consider network settings in which all agents can sign multiple contracts. Azevedo and Hatfield (2018) showed that stable outcomes exist in large, two-sided markets in which one side regards contracts as substitutes. Under those assumptions, however, we show that tree stability and stability coincide; as a result, our existence result for tree stability is equivalent to Azevedo and Hatfield’s (2018) existence result for stability under their assumptions, but also guarantees existence more generally—i.e., in the presence of arbitrary complementarities and network structures, and of bundling. More recently, Greinecker and Kah (2021) showed that pairwise stable outcomes exist in large one-to-one matching markets even in the presence of externalities, which are beyond the scope of this paper.

Previous work has also considered the impact of bundling on equilibrium in matching markets in the special cases in which all contracts and contract bundles are substitutable (Hatfield and Kominers, 2017), and in which all contracts are complementary (Rostek and Yoder, 2020). By contrast, our analysis applies regardless of the structure of agents’ preferences over contracts, and furthermore examines the relationship between bundling and coordination in terms of the core.[5] Allowing for general preferences is important because complementarities can arise due to bundling (Hatfield and Kominers, 2017).

Paper outline. The remainder of this paper is organized as follows. Section 2 introduces our basic model of matching in large networks without the possibility of bundling. Section 3 presents examples that illustrate the existence issues for matching markets. Section 4 develops our existence result for tree stability without bundling and relates it to Azevedo and Hatfield’s (2018) existence result. Section 5 incorporates the possibility of bundling into the model and compares tree stability to stability and the core. Section 6 is a conclusion. Appendix A provides some technical details related to

3. Those analyses required that input contracts be substitutable for each other, output contracts be substitutable for each other, but that input and output contracts be complementary to one another. By contrast, Rostek and Yoder (2020) allowed for general network topologies but assumed that all contracts are complementary—thereby ruling out all forms of substitutabilities. Baldwin and Klemperer’s (2019) results also give domains of preferences that can include forms of complementarity and substitutability, for which the existence of equilibrium is guaranteed with transferable utility. However, each of these domains place strong restrictions on the forms of substitutability and/or complementarity that can arise.

4. Che et al. (2019) considered a setting with finitely many firms on one side of the market.

5. While many stability concepts have been shown to coincide with the core in two-sided many-to-one matching markets (see, e.g., Echenique and Oviedo (2004)), they are different from the core in general (Blair, 1988).

the extension developed in Section 5. Appendix B discusses further relationships between tree stability, stability, and other stability concepts. Appendix C presents the proofs. The online appendixes contain additional results and examples.

2. MODEL

The basic framework is an extension of Azevedo and Hatfield's (2018) large-market two-sided matching model to networks. There is a continuum of agents, partitioned into finitely many types, that interact bilaterally with multiple counterparties via an exogenously specified set of contracts. We extend the basic model to incorporate the possibility of contingent contracts in Section 5.

2.1. *Agents, contracts, and preferences*

There is a finite set I of types. For each type $i \in I$, there is a (homogenous) mass $\theta^i > 0$ of agents of type i in the economy, parameterized by an interval $[0,\theta^i]$. The set of agents of type i is $\Theta_i = \{i\} \times [0,\theta^i]$, and hence the set of all agents is $\Theta = \bigcup_{i\in I} \Theta_i$. We let μ denote the measure on Θ whose restriction to each space Θ_i is given by the Lebesgue measure.

For each pair i,j of types, there is a finite set $X_{i,j} = X_{j,i}$ of contracts between an agent of type i and an agent of type j. A contract describes all aspects of a single interaction, including goods to be exchanged, services to be rendered, and transfers (Hatfield and Milgrom, 2005). To capture the possibility that agents can interact in multiple ways, we allow pairs of agents to sign multiple contracts with each other (Fleiner, 2003; Ostrovsky, 2008; Kominers, 2012; Hatfield and Kominers, 2017). We assume that contracts are type-specific: i.e., that $X_{i,j} \cap X_{i',j'} = \varnothing$ when $\{i,j\} \neq \{i',j'\}$. The set of contracts in which an agent of type i can engage is $X_i = \bigcup_{j\in I} X_{i,j}$.

Each type i has a utility function $u^i : \mathcal{P}(X_i) \to \mathbb{R}$ defined over the sets of contracts agents of the type can engage in. As is standard in matching theory, we consider agents' choices from sets of available contracts. Specifically, given a set $Y \subseteq X_i$ of contracts, let

$$C^i(Y) = \operatorname*{argmax}_{Z\subseteq Y} u^i(Z)$$

denote the family of utility-maximizing choices of an agent of type i when offered the contracts in Y.

2.2. *Outcomes*

In a market outcome, each agent participates in a set of contracts; these sets must be compatible across agents. Formally, a *matched type* consists of a type $i \in I$ and a set $Y \subseteq X_i$ of contracts. An outcome is described by a (measurable) set $M^i_Y \subseteq \Theta_i$ of agents of type i that participate in set Y for each matched type (i,Y) such that (1) each agent is associated to exactly one set of contracts, and (2) contracts are signed by equal masses of either counterparty.

Definition 1. *An* ***outcome*** *M consists of a measurable subset $M^i_Y \subseteq \Theta_i$ for each matched type (i,Y) such that*

- *(feasibility) the sets M^i_Y are disjoint and satisfy*

$$\bigcup_{Y\subseteq X_i} M^i_Y = \Theta_i$$

- *(reciprocity) for each pair of distinct types $i\neq j$ and each contract $x\in X_{i,j}$, we have that*

$$\mu\left(\bigcup_{Y\subseteq X_i|x\in Y} M^i_Y\right) = \mu\left(\bigcup_{Y\subseteq X_j|x\in Y} M^j_Y\right).$$

In Definition 1, feasibility requires that exactly one set of contracts be specified for each agent. In the reciprocity condition, $\bigcup_{Y\subseteq X_i|x\in Y} M^i_Y$ is the set of agents of type i that sign contract x. Hence, reciprocity requires that each contract be signed by equal masses of agents of each counterparty type—an analogue of the market-clearing condition from general equilibrium models.

2.3. *Tree stability*

We define tree stability by adapting Ostrovsky's (2008) definition to large markets. Intuitively, an outcome is tree-stable if there is no acyclic set (i.e., tree) of contracts that agents would desire to add given their current matches, possibly while dropping some existing contracts (Ostrovsky, 2008). Moving to the context of large markets, we allow for the possibility that blocking sets of contracts between measure-zero sets of agents persist in stable outcomes—following Azevedo and Hatfield (2018) and Greinecker and Kah (2021).

The first part of the definition of tree stability rules out deviations by individuals. Formally, individual rationality requires that no agents—except possibly those in a measure-zero set—want to unilaterally drop any contract.

Definition 2. *An outcome M is **individually rational** if we have that $\mu(M^i_Y)=0$ for each matched type (i,Y) with $Y\notin C^i(Y)$.*[6]

To define tree stability, we first introduce blocking sets of contracts with the shapes of arbitrary graphs and then restrict attention to acyclic graphs to define blocking trees. We consider graphs whose set of nodes is $\{1,2,\ldots,n\}$ and specify graphs by their sets of edges. Formally, a *graph with n nodes* is a family ν of two-element subsets of $\{1,2,\ldots,n\}$; the members of ν are called *edges*. A graph is a *tree* if there exists a unique path between each pair of nodes.[7]

Intuitively, a block of shape ν for an outcome M consists of agents $a_1,\ldots,a_n$, and a blocking set of contracts forming a subnetwork of shape ν, such that the participating agents find the blocking contracts desirable, possibly while dropping some of their existing contracts. Instead of identifying the participating agents $a_1,\ldots,a_n$ directly, we specify their matched types $(i_1,Y^1),\ldots,(i_n,Y^n)$ in M—which are sufficient to determine

6. This definition follows Roth (1984), Hatfield et al. (2013), and Azevedo and Hatfield (2018).

7. Formally, a *path between nodes j and k in a graph ν* is a sequence $1\leq j_1,\ldots,j_m\leq n$ of distinct integers such that $\{j_\ell,j_{\ell+1}\}\in\nu$ for all $0\leq\ell\leq m$, where $j_0=j$ and $j_{m+1}=k$.

whether the agents find a deviation desirable.[8] A block arises at M if, there are positive masses of agents of each of the participating matched types at M.

Definition 3. *Let ν be a graph with n nodes. A **block of shape** ν consists of*

- *a matched type (i_j, Y^j) for each node $1 \leq j \leq n$, and*
- *a contract $z_{j,k} = z_{k,j} \in X_{i_j, i_k}$ for each edge $\{j,k\} \in \nu$*

for which

- *[compatibility] for each node j, the contracts $(z_{j,k})_{\{j,k\} \in \nu}$ are distinct;*
- *[desirability] for each node j, writing*

$$Z^j = \left\{ z_{j,k} \mid \{j,k\} \in \nu \right\},$$

we have that $\varnothing \neq Z^j \subseteq X_{i_j} \setminus Y^j$ and that $Z^j \subseteq W^j$ for all $W^j \in C^{i_j}(Y^j \cup Z^j)$.

*Such a block **arises** at an outcome M if $\mu\left(M^{i_j}_{Y^j}\right) > 0$ for all j.*

Here, compatibility requires that no agent participate in the same contract multiple times during a block. As in Hatfield et al. (2013), desirability requires the blocking contracts not be among participating agents' existing contracts, and that the blocking contracts be chosen at every selection from participating agents' choices among their existing and blocking contracts.[9] Intuitively, with limited commitment power, this desirability condition ensures that agents have no incentives to renege on blocking contracts.

We next define three stability concepts: pairwise stability, tree stability, and stability. These concepts are the large-market limits of the corresponding concepts in finite markets (see, e.g., Azevedo and Hatfield (2018) and Greinecker and Kah (2021)).

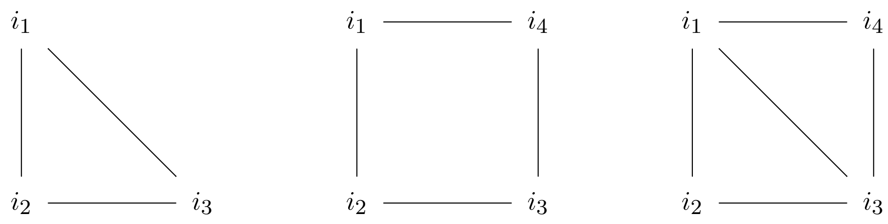
Definition 4. *An outcome is:*

- ***pairwise-stable** if it is individually rational and no block of the shape of a graph with two nodes and one edge arises.*
- ***tree-stable** if it is individually rational and no block of shape ν arises for any tree ν.*
- ***stable** if it is individually rational and no block of any shape arises.[10]*

8. Note that the matched types $(i_1, Y^1), \ldots, (i_n, Y^n)$ involved in a block may not be distinct. It is important to allow this possibility, as multiple agents of the same matched type may be needed to take on different positions in a block. For example, if production requires two units of a good, then a block may need to involve two unmatched producers, who may be of the same type.

9. Since desirability allows agents to retain some existing contracts, blocks do not generally give rise to core blocks in the sense of cooperative game theory (Sotomayor, 1999; Echenique and Oviedo, 2006); since desirability requires more than just that the blocking contracts improve utility, core blocks do not generally give rise to blocks either (Blair, 1988; Echenique and Oviedo, 2006).

10. While we do not formally allow for multiple contracts between pairs of agents in a block, this restriction does not affect the definition of stability in our large-market model—as we show in Appendix B.1. Thus, our definition of stability coincides with Azevedo and Hatfield's (2018) definition



(a) Example 1. (b) Example 2. (c) Example 3.

FIGURE 1
Contracts in our examples. In each case, an edge between types i_j and i_k corresponds to a contract $x_{j,k}$.

In Appendix B.2, we show that tree stability can equivalently be defined using a weaker desirability condition that only requires that the block lead to utility improvements for all agents, rather than requiring that the participating agents always choose all of the blocking contracts (when given access to their existing contracts).

It is clear that stability implies tree stability and that tree stability implies pairwise stability. In markets in which agents are only willing to sign one contract, pairwise stability, tree stability, and stability all coincide. But, in general, tree stability is strictly weaker than stability and pairwise stability is strictly weaker than tree stability, as we show in the next section.

3. ILLUSTRATIVE EXAMPLES

In this section, we provide examples that illustrate the existence issues for stability concepts. We first present an example that illustrates the issues for pairwise stability (and hence tree stability and stability) in finite markets. We then present examples that contrast pairwise stability, tree stability, and stability—including an example that demonstrates that stable outcomes may not exist, even in large markets.

In all of the examples, we specify types' preferences over sets of contracts that are preferable to the outside option of remaining unmatched; utility functions can be taken to be arbitrary representations of those preferences.

Our first example is the "roommates problem" from Gale and Shapley (1962) discussed in the introduction, in which there are no pairwise-stable outcomes in finite markets.

Example ***1 (Nonexistence of pairwise-stable outcomes in finite markets—Gale and Shapley, 1962*** *As depicted in Figure 1(a), there are three types, and one contract between each pair of types. Each agent desires to engage in one contract, and types' preferences are given by*

$$i_1 : \{x_{1,2}\} \succ \{x_{1,3}\} \succ \varnothing \qquad i_2 : \{x_{2,3}\} \succ \{x_{1,2}\} \succ \varnothing \qquad i_3 : \{x_{1,3}\} \succ \{x_{2,3}\} \succ \varnothing.$$

in their two-sided context without indifferences. Furthermore, unlike in finite markets, implementing blocks leads to utility improvements for all participating agents with probability 1 (Vocke, 2022).

Intuitively, each agent would like to have a roommate, and each agent of type i_1 (resp., type i_2, type i_3) would prefer to have a roommate of type i_2 (resp., type i_3, type i_1).

In a finite market in which there is only one agent of each type, each outcome is blocked by a set consisting of a single contract. Indeed, it is not possible to match all three agents, and the contract $x_{1,3}$ (resp., $x_{1,2}$, $x_{2,3}$) blocks any outcome in which the agent of type i_1 (resp., type i_2, type i_3) is unmatched. In particular, no pairwise stable outcomes exist; hence, no tree-stable or stable outcomes exist either.

On the other hand, in a large market with a unit mass of agents of each type, there are outcomes in which all agents are matched, namely the outcomes in which $\frac{1}{2}$ mass of agents of type i_1 are matched to each of types i_2 and i_3, and $\frac{1}{2}$ mass of agents of type i_2 and i_3 are matched with one another.[11] *These outcomes are pairwise-stable, and in fact even tree-stable and stable.*

Example 1 illustrates that in finite markets that lack a two-sided structure, pairwise-stable outcomes do not generally exist. As Example 2.7 in Roth and Sotomayor (1990) shows, the same issue arises in two-sided many-to-one matching markets when there are complementarities. Hence, to guarantee existence for any concept that refines pairwise stability, one needs either to impose conditions on the structure of the market and on preferences—an approach taken by much of the previous literature—or consider a continuum of agents—as we do.

In Example 1, pairwise stability, tree stability, and stability coincide as each agent wants to participate in only one interaction. Our remaining examples illustrate the distinctions between pairwise stability, tree stability, and stability. Our second example shows that tree stability is a strictly stronger condition than pairwise stability in general.

Example *2* ***(Pairwise stability versus tree stability)*** *As depicted in Figure 1(b), there are four types and four contracts. There is a unit-mass continuum of agents of each type. The types' preferences are given by*

$$i_1 : \{x_{1,2}\} \succ \{x_{1,4}\} \succ \varnothing \qquad i_2 : \{x_{2,3}\} \succ \{x_{1,2}\} \succ \varnothing$$
$$i_3 : \{x_{3,4}\} \succ \{x_{2,3}\} \succ \varnothing \qquad i_4 : \{x_{1,4}, x_{3,4}\} \succ \varnothing.$$

Intuitively, this example is a version of the roommates' problem (Example 1) in which type i_4 acts as an intermediary between types i_1 and i_3. Note that the example can be regarded as a two-sided many-to-one matching market in which i_2 and i_4 comprise one side of the market and i_1 and i_3 comprise the other side.

The outcome in which all agents of types 2 and 3 are matched (via the contract $x_{2,3}$) is pairwise-stable. Indeed, the only blocks that arise at this outcome require multiple contracts: specifically, the contracts $x_{1,4}$ and $x_{3,4}$. As the contracts $x_{1,4}$ and $x_{3,4}$ comprise a tree, the outcome is not tree-stable (hence not stable either).[12]

11. An example of such an outcome is the outcome defined by

$$M^{i_1}_{\{x_{1,2}\}} = \{i_1\} \times \left[0, \frac{1}{2}\right) \qquad M^{i_2}_{\{x_{2,3}\}} = \{i_2\} \times \left[0, \frac{1}{2}\right) \qquad M^{i_3}_{\{x_{1,3}\}} = \{i_3\} \times \left[0, \frac{1}{2}\right)$$
$$M^{i_1}_{\{x_{1,3}\}} = \{i_1\} \times \left[\frac{1}{2}, 1\right] \qquad M^{i_2}_{\{x_{1,2}\}} = \{i_2\} \times \left[\frac{1}{2}, 1\right] \qquad M^{i_3}_{\{x_{2,3}\}} = \{i_3\} \times \left[\frac{1}{2}, 1\right].$$

12. Despite the presence of pairwise-stable outcomes that are not tree-stable, tree-stable outcomes exist. In fact, there is an essentially unique tree-stable outcome. Specifically, the tree-stable outcomes

Example 2 illustrates that pairwise-stable outcomes can be vulnerable to simple deviations that tree stability rules out. However, tree stability and stability coincide in Example 2. Our final example illustrates the distinction between tree stability and stability for more general network topologies. It also highlights that stable outcomes can fail to exist even in large markets, as shown by Azevedo and Hatfield (2018).

Example ***3 (Tree stability versus stability)*** *As depicted in Figure 1(c), there are four types and five contracts. There is a unit-mass continuum of agents of each type. The types' preferences are given by*

$$i_1 : \{x_{1,2}, x_{1,4}\} \succ \{x_{1,3}, x_{1,4}\} \succ \varnothing \qquad i_2 : \{x_{1,2}\} \succ \{x_{2,3}\} \succ \varnothing$$
$$i_3 : \{x_{1,3}, x_{3,4}\} \succ \{x_{2,3}\} \succ \varnothing \qquad i_4 : \{x_{1,4}, x_{3,4}\} \succ \varnothing.$$

Intuitively, agents of types i_1, i_3, and i_4 would like to match with each other, i.e., simultaneously sign the contracts $x_{1,3}$, $x_{3,4}$, and $x_{1,4}$. However, agents of types i_1 and i_3 also have the possibility of matching with agents of type i_2: agents of type i_1 would like to do so only if they can also match with an agent of type i_4, while agents of type i_3 would like to do so only if they are unable to match with agents of types i_1 and i_4.

There is an essentially unique pairwise stable outcome. Specifically, the pairwise-stable outcomes are the outcomes in which mass 1 of agents of types i_2 and i_3 are matched with each other, and mass 0 of agents of types i_1 and i_4 are matched. [13] *The pairwise stable outcomes are tree-stable as well.*

However, the pairwise/tree-stable outcomes are not stable. Indeed, a block of the shape of a cycle—involving the contracts $x_{1,3}$, $x_{3,4}$, and $x_{1,4}$—arises. Hence, no stable outcomes exist—even in large markets.

Example 3 shows that nonexistence can arise—even in large markets—if agents can form blocks of shapes that involve cycles. Thus, there is a fundamental issue with analyzing markets with complex preferences using stability. Conversely, we show in the next section that tree-stable outcomes are guaranteed to exist in large markets with arbitrary preferences and network structures.

4. EXISTENCE

4.1. *Existence of pairwise stable outcomes*

Our first main result shows that pairwise stable outcomes always exist in large markets for arbitrary preferences and network structures.

in this example are the outcomes in which $\frac{1}{2}$ mass of agents of type 1 sign each of $x_{1,2}$ and $x_{1,4}$ (resp., $x_{1,2}$ and $x_{2,3}$, $x_{2,3}$ and $x_{3,4}$), $\frac{1}{2}$ mass of agents of type 4 sign both $x_{1,4}$ and $x_{3,4}$, and $\frac{1}{2}$ mass of agents of type 4 remain unmatched. These outcomes are even stable.

13. Indeed, it is straightforward to verify that such outcomes are pairwise-stable. Conversely, in any individually rational outcome, only zero mass of agents can sign $x_{1,2}$, as if a positive mass of agents were to sign $x_{1,2}$, a larger mass of $x_{1,4}$ would have to be signed than $x_{1,3}$ in aggregate (in light of type i_1's preferences)—an occurrence that is incompatible with the preferences of types i_3 and i_4. However, any outcome in which a positive mass of agents signs contract $x_{1,3}$ is not pairwise-stable, as a block involving the single contract $x_{1,2}$ would arise. Given the preferences of types i_3 and i_4, it follows that no outcome in which a positive mass of agents signs contract $x_{1,4}$ or $x_{3,4}$ can be pairwise-stable. Hence, the only contract that a positive mass of agents can sign in a pairwise-stable outcome is $x_{2,3}$.

Theorem 1. *Pairwise stable outcomes exist.*

This existence result is a new result even in the two-sided case. Most stability concepts are refinements of pairwise stability; thus, this result delivers the first existence result for a stability concept in a matching market with general preferences. While pairwise stability is a standard concept in two-sided one-to-one markets, more complex blocks than single-contract deviations arise in many-to-many matching markets with complementarities. In the sequel, we give an existence result for a refinement of pairwise stability that allows agents to sign multiple contracts simultaneously in deviations.

4.2. *Existence of tree-stable outcomes*

Our first main result shows that tree-stable outcomes always exist in large markets for arbitrary preferences and network structures.

Theorem 2. *Tree-stable outcomes exist.*

Unlike previous results for other solution concepts (such as those of Azevedo and Hatfield (2018) and Che et al. (2019)), no conditions on preferences (such as substitutability or unit demand) or the structure of the network are needed.

To prove Theorem 2, we adapt Azevedo and Hatfield's (2018) strategy for showing the existence of stable outcomes in large, two-sided matching markets in which one side of the market has substitutable preferences. Relative to Azevedo and Hatfield's (2018) argument, our proof needs to allow for indifferences, and for arbitrary preferences and network topologies. The argument proceeds in four steps. First, we construct an aggregate choice correspondence over masses of contracts by extending Azevedo and Hatfield's (2018) construction to allow for indifferences.[14] Second, we use the aggregate choice correspondence to construct a generalized Gale–Shapley operator—adapting Che et al.'s (2021) construction to a continuum network setting and Hatfield and Kominers's (2012) construction to a continuum setting with indifferences. Third, we apply Kakutani's Fixed Point Theorem to show that the generalized Gale–Shapley operator must have a fixed point. Fourth, we show that each fixed point gives rise to tree-stable outcomes.

The fourth step of the argument is the place at which Azevedo and Hatfield (2018) used a substitutability assumption in their two-sided context to show that, for any block, some participating agent must be willing to sign a single blocking contract on its own. We instead use the restriction to blocks of the shape of trees to avoid having to require any substitutability condition for preferences. Intuitively, for any blocking tree, there are some agents who participate in only one contract in the block—namely the agents corresponding to the "leaves" of the tree. By construction, each such agent is willing to sign a single blocking contract on its own.

4.3. *Relationship to Azevedo and Hatfield (2018)*

We now relate Theorem 2 to Azevedo and Hatfield's (2018) existence result for large, two-sided matching markets. Specifically, we show that tree stability and stability coincide

14. Greinecker and Kah (2021) made a related construction in a one-to-one setting, adapting a earlier construction of Jagadeesan (2017).

under Azevedo and Hatfield's (2018) sufficient condition for the existence of stable outcomes. Hence, Theorem 2 offers a generalization of Azevedo and Hatfield's (2018) result that applies in the presence of arbitrary forms of complementarities and general network structures.

To state Azevedo and Hatfield's (2018) result, we first recall the definition of substitutability from the matching literature (Kelso and Crawford, 1982; Hatfield and Milgrom, 2005). Formally, type i's preferences are *substitutable* if for all sets $Y \subseteq Y' \subseteq X_i$ of contracts and all choices $Z \in C^i(Y)$, there exists a choice $Z' \in C^i(Y')$ such that $Z' \cap Y \subseteq Z$. Intuitively, substitutability requires that expanding the set of available contracts from Y to Y' not make any contract in Y more desirable. Due to the possibility of indifferences, we technically impose that for each utility-maximizing choice from Y, there exists a utility-maximizing choice from Y' that does not include any previously unchosen contract from Y (Hatfield et al., 2019).[15]

A market is two-sided if types can be divided into buyer and seller types so that each contract is between a buyer type and a seller type. Formally, for disjoint sets $B, S \subseteq I$ of types, we say that the network is *two-sided with buyer types B and seller types S* if $I = B \cup S$, $X_{b,b'} = \varnothing$ for all $b, b' \in B$, and $X_{s,s'} = \varnothing$ for all $s, s' \in S$.

Azevedo and Hatfield (2018) showed that stable outcomes exist in two-sided markets if all seller types have substitutable preferences. Under Azevedo and Hatfield's (2018) conditions, stability and tree stability in fact coincide.

Proposition 1. *Suppose that the network is two-sided with buyer types B and seller types S. If each seller type $s \in S$ has substitutable preferences, then an outcome is stable if and only if it is tree-stable.*

To understand Proposition 1, note that stable outcomes are always tree-stable. For the converse direction, we show that when sellers have substitutable preferences, any block can be transformed into a block in which each participating seller signs only one blocking contract—i.e., a blocking tree.

Proposition 1 tells us that Theorem 2 has the same implications as Azevedo and Hatfield's (2018) result under the conditions under which Azevedo and Hatfield's (2018) result applies.[16] But Theorem 2 is more general: it shows that tree-stable outcomes exist in two-sided markets in the presence of arbitrary forms of complementarities, as well as demonstrating existence for arbitrary network structures. And working with tree stability is essential to obtaining the generalization to arbitrary preferences (see, e.g., Example 3).

5. BUNDLING, CONTINGENCIES, AND MULTILATERAL CONTRACTS

In this section, we develop an extension to address a conceptual issue with modeling matching markets using a continuum framework. In practice, interactions among small groups of agents are often non-anonymous—especially in matching markets. For example, firms often form long lasting relationships with suppliers and resellers. However, a

15. We use the "expansion substitutability" condition from Appendix A of Hatfield et al. (2019). See also Appendix A of Fleiner, Jagadeesan, Jankó, and Teytelboym (2019) and Section 6 of Che et al. (2021). Azevedo and Hatfield (2018) did not have to deal with this issue as they ruled out indifferences.

16. In Online Appendix D, we show that Proposition 1 extends to vertical supply chain settings—leading to a supply chain extension of Azevedo and Hatfield's (2018) existence result.

consequence of modeling the set of agents of a given type as a continuum is the anonymity (Aumann, 1964) or symmetry (Milnor and Shapley, 1978) of agents in that set.

To capture the coordination possibilities present in smaller markets within our large-market framework, we extend our model to allow contracts to be made contingent on each other. Technically, we allow certain networks of contracts to be bundled together. Intuitively, bundling some contracts in an outcome or a block makes them contingent on each other—increasing the ability of participating agents to coordinate, and, in effect, allowing them to form relationships that consist of multiple contracts. The possibility of blocking contracts contingent on each other also lets groups of agents coordinate on some more complex deviations involving cycles. Nevertheless, the possibility of making executed contracts contingent on each other ensures that existence carries over for a suitably adapted version of tree stability.

We first formally define contract bundles. We then extend the definition of tree stability and the existence result for tree stability to settings in which agents can participate in contract bundles. Last, we explain how the bundling can allow agents to achieve full coordination by showing how the possibility of bundling affects the relationship between tree stability, stability, and the core.

5.1. *Model of bundling*

A contract bundle involving n agents specifies the types of the participating agents as well as the contracts between the agents that are part of the contract bundle. While we formally assume that contract bundles are comprised of bundles of bilateral contracts, our model with bundling can actually capture settings with multilateral contracts as well.[17]

Definition 5. *Let $n \geq 2$. A **contract bundle** χ **involving** n **agents** consists of types $\chi(1),\ldots,\chi(n)$, and, for each $1 \leq \ell, m \leq n$, a set $\chi_{\ell,m} = \chi_{m,\ell} \subseteq X_{\chi(\ell),\chi(m)}$ of contracts such that:*

- *[compatibility] for each index ℓ, we have that $\chi_{\ell,\ell} = \varnothing$ and the sets $(\chi_{\ell,m})_{1 \leq m \leq n}$ are disjoint;*
- *[connectedness] for all sets L_1, L_2 with $L_1 \cup L_2 = \{1,2,\ldots,n\}$, there exist $\ell \in L_1$ and $m \in L_2$ such that $\chi_{\ell,m} \neq \varnothing$.*

As contract bundles and blocks each correspond to concepts of coordination among groups of agents, the definition of a contract bundle (Definition 5) is similar to the definition of a block (Definition 3). Connectedness simply requires that a contract bundle be irreducible, in that it cannot be decomposed into two separate contract bundles between disjoint groups of agents. Intuitively, we do not allow disconnected contracts to be made contingent on one another.

For each n, there is a set $\mathcal{X}[n]$ of feasible contract bundles involving n agents. While a contract bundle specifies an ordering among participants, we assume that the ordering

17. Indeed, by arbitrarily choosing a leader out of the group of agents that can participate in a multilateral contract, one can decompose the multilateral contract into a set of bilateral contracts between the leader and each of the other participants. The contract bundle consisting of all of these bilateral contracts then recovers the original multilateral contract.

does not affect which contract bundles are feasible.[18] To ensure that there is a finite number of feasible contract bundles, we also assume that $\mathcal{X}[n]=\varnothing$ for sufficiently large n—that is, that very large groups of agents cannot form contract bundles.

Note that contract bundles can involve multiple agents of the same type in different ways. Economically, this feature captures the possibility, for example, that a contract bundle include several similar firms that produce different products. To distinguish between the ways in which an agent can participate in a contract bundle, we consider the roles that an agent can play in a contract bundle. Formally, a *role* is a pair (χ,ℓ), where $\chi\in\mathcal{X}[n]$ and $1\leq\ell\leq n$. Such a role corresponds to the involvement of the ℓth agent in χ: in such a role, an agent of type $\chi(\ell)$ participates in the set $\chi_\ell=\bigcup_{m=1}^{n}\chi_{\ell,m}\subseteq X_{\chi(\ell)}$ of contracts. For each type i, we let $\mathcal{R}_i$ denote the set of roles for agents of type i.

In an outcome with bundling, each agent participates in a set of roles; these roles must be compatible across agents. Analogously to a matched type, we define a *bundle-matched type* to be a pair $(i,\mathcal{Y})$, where $\mathcal{Y}\subseteq\mathcal{R}_i$. An outcome with bundling consists of a (measurable) set $\mathcal{M}^i_{\mathcal{Y}}$ of agents of type i that participate in set $\mathcal{Y}$ of roles for each bundle-matched type $(i,\mathcal{Y})$ such that (1) each agent is associated to exactly one set of roles, and (2) for each contract bundle, equal masses of agents participate in each constituent role.

Definition 1′. *An **outcome with bundling** (or, more simply, a **bundled outcome**) $\mathcal{M}$ consists of a measurable subset $\mathcal{M}^i_{\mathcal{Y}}\subseteq\Theta_i$ for each bundle-matched type $(i,\mathcal{Y})$ such that*

- *(feasibility) the sets $\mathcal{M}^i_{\mathcal{Y}}$ are disjoint and satisfy*

$$\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_i}\mathcal{M}^i_{\mathcal{Y}}=\Theta_i$$

- *(reciprocity) for each n and each feasible contract bundle $\chi\in\mathcal{X}[n]$, we have that*

$$\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(1)}|(\chi,1)\in\mathcal{Y}}\mathcal{M}^{\chi(1)}_{\mathcal{Y}}\right)=\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(2)}|(\chi,2)\in\mathcal{Y}}\mathcal{M}^{\chi(2)}_{\mathcal{Y}}\right)=\cdots=\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(n)}|(\chi,n)\in\mathcal{Y}}\mathcal{M}^{\chi(n)}_{\mathcal{Y}}\right).$$

Here, in the reciprocity condition,

$$\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(\ell)}|(\chi,\ell)\in\mathcal{Y}}\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}$$

is the set of agents of type $\chi(\ell)$ that participate in role (χ,ℓ). Hence, reciprocity requires that equal masses of agents participate in each of the constituent roles of each contract bundle.

5.2. *An illustrative example*

The following example illustrates how the contigencies entailed by bundling can affect both the tree stability of outcomes and the existence of stable outcomes. Outcomes that

18. More precisely, we assume that permuting types does not affect which contract bundles are feasible. That is, we assume that if $\chi\in\mathcal{X}[n]$ is a feasible contract bundle and $\sigma:\{1,\ldots,n\}\to\{1,\ldots,n\}$ is a permutation, then the contract bundle χ^σ involving n agents defined by $\chi^\sigma(\ell)=\chi(\sigma^{-1}(\ell))$ and $\chi^\sigma_{\ell,\ell'}=\chi_{\sigma^{-1}(\ell),\sigma^{-1}(\ell')}$ is also feasible (i.e., $\chi^\sigma\in\mathcal{X}[n]$).

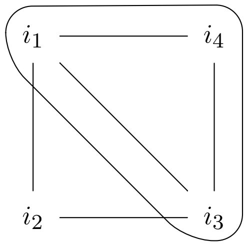


FIGURE 2

Feasible contract bundles in Example 4. Contracts are as in Example 3 (see Figure 1(c)). Each contract is feasible as a contract bundle; the trilateral contract bundle χ among types i_1, i_3, and i_4 involving contracts $x_{1,3}$, $x_{1,4}$, and $x_{3,4}$ is also feasible (as are permutations thereof).

are tree-stable without bundling can fail to be tree-stable with bundling due to the presence of new blocks, and bundled outcomes can be stable—even if no stable outcomes exist without bundling.

Example ***4 (Bundling in Example 3)*** *This example extends Example 3 by incorporating the possibility of bundling. There are 4 types with a unit mass of agents of each type, and contracts and preferences are as in Example 3: contracts correspond to the edges in Figure 1(c), and the types' preferences over sets of contracts are:*

$$i_1:\{x_{1,2},x_{1,4}\}\succ\{x_{1,3},x_{1,4}\}\succ\varnothing \qquad i_2:\{x_{1,2}\}\succ\{x_{2,3}\}\succ\varnothing$$
$$i_3:\{x_{1,3},x_{3,4}\}\succ\{x_{2,3}\}\succ\varnothing \qquad i_4:\{x_{1,4},x_{3,4}\}\succ\varnothing.$$

Each contract is feasible as a bilateral contract bundle. We also allow the contracts $x_{1,3}$,$x_{1,4}$, and $x_{3,4}$ to be bundled to form a multilateral contract bundle χ, see Figure 2.

The essentially unique tree-stable outcome in Example 3 (i.e., the outcomes in which mass 1 of agents of types i_2 and i_3 are matched with each other, and mass 0 of agents of types i_1 and i_4 are matched)[19] is no longer tree-stable with bundling. Indeed, while without bundling, there was only a cyclic block of this outcome involving contracts $x_{1,3}$,$x_{1,4}$, and $x_{3,4}$; with bundling, the contract bundle χ comprises a block.

Without bundling, there are no stable outcomes. However, with bundling, there is an essentially unique stable bundled outcome, in which mass 1 of each of types i_1, i_3, and i_4 sign the contract bundle χ.[20] Intuitively, the fact that $x_{1,3}$ and $x_{1,4}$ are bundled makes them contingent on each other, and therefore prevents agents of type i_1 from dropping contract $x_{1,3}$ and signing contract $x_{1,2}$ while maintaining contract $x_{1,4}$. In fact, with

19. An example of a representation of such an outcome with bundling is the bundled outcome defined by

$$\mathcal{M}^{i_1}_{\varnothing}=\{i_1\}\times[0,1] \qquad \mathcal{M}^{\chi^{2,3}(\ell)}_{\{(\chi^{2,3},\ell)\}}=\{\chi^{2,3}(\ell)\}\times[0,1] \text{ for } 2\leq\ell\leq3 \qquad \mathcal{M}^{i_4}_{\varnothing}=\{i_4\}\times[0,1],$$

where $\chi^{2,3}$ is the contract bundle involving two agents defined by the types i_2,i_3 and the contract $x_{2,3}$.

20. An example of such an bundled outcome is one defined by

$$\mathcal{M}^{i_2}_{\varnothing}=\{i_2\}\times[0,1] \qquad \mathcal{M}^{\chi(\ell)}_{\{(\chi,\ell)\}}=\{\chi(\ell)\}\times[0,1] \text{ for } 1\leq\ell\leq3.$$

bundling, the tree-stable bundled outcomes are stable (and in the core)—unlike the case without bundling.

In the remainder of this section, we extend our existence result for tree stability to settings with bundling. We then show that tree stability essentially coincides with stability and the core if all contract bundles among small groups of agents are feasible.

5.3. *Stability concepts and existence*

The definitions of tree stability and stability can be extended to the setting with bundling by using hypergraphs instead of graphs; see Appendix A for the technical details. As the setting without bundling is a special case of the setting with bundling, stable bundled outcomes do not exist in general. Nevertheless, the existence result for tree stability continues to hold in settings with bundling.

Theorem 2′. *Tree-stable bundled outcomes exist.*

The proof of Theorem 2′ follows a similar approach to the proof of Theorem 2. The key conceptual challenge relative to that argument is that the extended model allows for multilateral contracting via contract bundles. We deal with this challenge by working with an auxiliary two-sided matching market in which agents match with contract bundles.

We next explain the relationship between tree stability and stability in settings with bundling, and discuss implications for the existence of stable bundled outcomes.

Example 3 shows that stability is generally strictly stronger than tree stability. However, it turns out that tree stability coincides with stability when all small groups of agents can coordinate. Formally, we suppose that all contract bundles among at most N agents are feasible for some N.

Theorem 3. *For each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then every tree-stable bundled outcome is stable.*

To understand Theorem 3, note that when enough contract bundles are feasible, blocks that involve cycles can be bundled—as in Example 4—making tree stability coincide with stability. Note that this intuition applies in finite markets; indeed, Theorem 3 actually holds in finite markets as well, and is a new result even in that context. In the context of large markets, the crux of the proof is to establish that for some finite N, the feasibility of contract bundles involving up to N agents is sufficient to capture each relevant cycle within a contract bundle. For example, in Example 4, tree stability and stability coincide when trilateral contract bundles are feasible ($N=3$).

When tree stability coincides with stability, the existence of tree-stable bundled outcomes (Theorem 2′) implies the existence of stable outcomes.

Corollary 1. *For each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then stable bundled outcomes exist.*

Corollary 1 shows that complementarities do not obstruct the existence of stable outcomes when enough contract bundles are feasible. Intuitively, the possibility of bundling provides agents with enough contingencies in their interactions to limit deviations from an outcome and therefore ensure the existence of stable outcomes in large markets. For example, in Example 4, bundling allows agent of types i_3 and i_4 to, in effect, make contract $x_{1,4}$ contingent on $x_{1,3}$—thereby preventing a deviation between agent of types i_1 and i_2.

5.4. *Contingencies and coordination power*

We now show how the contigencies entailed by bundling provide agents with coordination power by connecting tree stability to the core. We begin by defining core outcomes in our model—adapting the standard definition from cooperative game theory. Intuitively, the core differs from (tree) stability in that members of blocking coalitions cannot retain existing contracts with agents outside the coalition, and in that agents can commit to deviations that are not individually rational.

Formally, the definition of the core is similar to the definition of stability, but with three differences. First, core blocks specify all contracts among agents involved in a block instead of merely the new ones, so agents cannot retain contracts with agents outside the blocking coalition. Second, the desirability condition is replaced by a "preferability" condition, which requires that each agent prefer the set of blocking contracts in which she is involved to the set of contracts the outcome prescribes for her. Third, core outcomes are not required to be individually rational.

Definition 6. *Let ν be a graph with n nodes. A **strict** (resp., **weak**) **core block of shape** ν consists of*

- *a matched type (i_j, Y^j) for each node $1 \le j \le n$, and*
- *a contract $z_{j,k} = z_{k,j} \in X_{i_j,i_k}$ for each edge $\{j,k\} \in \nu$*

for which

- *[compatibility] for each node j, the contracts $(z_{j,k})_{\{j,k\}\in\nu}$ are distinct;*
- *[preferability] for each node j, writing*

$$Z^j = \{z_{j,k} \mid \{j,k\} \in \nu\},$$

we have that $u^{i_j}(Z^j) > u^{i_j}(Y^j)$ (resp., $u^{i_j}(Z^j) \ge u^{i_j}(Y^j)$ with strict inequality for some j).

An outcome is in the weak *(resp.,* strict*)* core *if no strict (resp., weak) core block arises.*

While we do not formally allow for multiple contracts between pairs of agents in a core block, this restriction does not affect the definition of the core in our model—as we show in Appendix B.1. Thus, our definition of the core coincides with the standard one (see, e.g., Azevedo and Hatfield (2018) for an analogous definition for large matching markets).

As core blocks only depend on agents' utilities in an outcome, they are independent of how contracts are bundled. To compare tree stability (which depends on bundling) to the core (which does not), we introduce a concept of equivalence up to bundling between bundled outcomes and (unbundled) outcomes. Intuitively, a bundled outcome is equivalent up to bundling to an outcome if for each agent, the outcome and the bundled outcome prescribe the same set of contracts up to bundling.

Definition 7. *An outcome M and a bundled outcome $\mathcal{M}$ are* ***equivalent up to bundling*** *if for all bundle-matched types $(i,\mathcal{Y})$, letting $Y = \bigcup_{(\chi,\ell)\in\mathcal{Y}} \chi_\ell$ denote the set of contracts involved in roles in $\mathcal{Y}$, we have that $\mathcal{M}^i_{\mathcal{Y}} \subseteq M^i_Y$.*

We now show that tree stability is essentially equivalent to the core if all contract bundles involving small groups of agents are feasible. More precisely, we show that in this case, tree stability is equivalent up to bundling to a solution concept between the strict and weak cores.

Theorem 4. *For each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then:*

(a) Each tree-stable bundled outcome is equivalent up to bundling to a weak core outcome.
(b) Each strict core outcome is equivalent up to bundling to a tree-stable bundled outcome.

Theorem 4 shows that if all contract bundles involving small groups of agents are feasible, then tree stability implies coordination on (weak) core outcomes—hence, in particular, that stable outcomes are weakly Pareto-efficient. Moreover, any (strict) core outcome can be supported under tree stability. Thus, when enough contract bundles are feasible, agents in effect have full coordination power under tree stability. In contrast, when no contract bundles are feasible, agents have limited ability to coordinate deviations across contracts under tree stability. The case in which only some of the small contract bundles are feasible interpolates between these two extremes—in effect, giving some groups of agents, but not others, the ability to coordinate.

The intuition behind Theorem 4(a) is similar to the intuition behind Theorem 3. Indeed, note that when enough contract bundles are feasible, the contracts involved in strict core blocks can be bundled together to form a contract bundle that may be desirable to agents even when the involved contracts would not be individually rational (without bundling). That is, making the contracts involved in a block contingent on each other improves agents’ coordination power enough to achieve core outcomes.

To understand Theorem 4(b), note that if enough contract bundles are feasible, then any outcome is equivalent up to bundling to a bundled outcome in which each agent participates in at most one contract. Since each agents’ contracts are then all contingent on each other, agents do not have the option to retain some but not all of their existing contracts when deviating.[21] The intuitions behind both parts of Theorem 4 apply in finite markets; indeed, Theorem 4 actually holds in finite markets as well, and is a new result even in that context.

21. In Online Appendix E, we show by example that the conclusion of Theorem 4(a) (resp. Theorem 4(b)) does not hold for the strict core (resp. weak core).

Theorem 4 also has a technical consequence: it follows from Theorems $2'$ and 4(a) that *weak core outcomes always exist in large markets*—giving an alternative proof of a result of Azevedo and Hatfield (2018) (see Theorem 2 in Azevedo and Hatfield (2018)).[22]

We prove Theorems 3 and 4(a) via a unified approach (see Appendix B.2). The argument actually shows that if all small contract bundles are feasible, then every tree-stable bundled outcome is *strongly group stable* (in the sense of Hatfield et al. (2013)). In fact, we also show in Appendix B.2 that the conclusion of Theorem 4(b) can be strengthened to give an equivalence to strongly group stable bundled outcomes.

6. CONCLUSION

While matching models are well-suited to analyzing markets with highly differentiated contracts, pairwise stable outcomes do not generally exist in the presence of real-world features such as complementarities and complex network topologies. To address the foundations of matching markets with general preferences and network topologies, we develop a model of matching in networks with a continuum of agents. We show that tree-stable outcomes are guaranteed to exist for arbitrary preferences and network structures. We also extend our framework by allowing certain subnetworks of contracts to be made contingent on each other, thereby capturing the possibility of non-anonymous relationships within a large market. These results suggest that tree stability is an appropriate solution concept for analyzing matching markets with complex preferences and network topologies.

Our focus on acyclic deviations has a noncooperative basis in the context of large markets, which Vocke (2023) develops. The idea is that if agents coordinate on deviations by iteratively making offers of contract bundles to (groups of) other agents but can only target offers at random agents of a given type, then agents will never find cyclic deviations. Here, the inability to target offers at specific other agents simply reflects the anonymity or symmetry of agents in a continuum. In the extended model with bundling, however, a degree of non-anonymity is allowed because the offer of a contract bundle can entail several contingent contracts with the offeror, as well as contracts with other offer recipients. This analysis provides a microfoundation for tree stability in large markets.

Our work leaves several directions for future research. First, externalities could be incorporated into our analysis, as in Greinecker and Kah (2021).[23] Second, the properties of tree-stable outcomes in our model could be investigated. Third, efficient algorithms to compute tree-stable outcomes could be explored.

A. FORMAL DEFINITION OF (TREE) STABILITY FOR BUNDLED OUTCOMES

In this appendix, we extend the definitions of tree stability and stability to settings with bundling. The key technical difference is that due to the presence of multilateral contracts, blocks take the shape of hypergraphs instead of graphs, and we therefore need to consider an appropriate extension of the concept of trees.

We first define agents' choice correspondences over sets of roles. Role (χ,ℓ) corresponds to the set χ_ℓ of contracts, and hence each set $\mathcal{Y}$ of roles corresponds to a set

$$\tau(\mathcal{Y}) = \bigcup_{(\chi,\ell)\in\mathcal{Y}} \chi_\ell$$

22. In Online Appendix E, we show by example that strict core outcomes do not exist in general.

23. Recently, Carmona and Laohakunakorn (2023) have made progress in this direction.

of contracts. An agents' utility of a set $\mathcal{Y}$ of roles is her utility of the corresponding set $\tau(\mathcal{Y})$ of contracts—except if some of the roles in $\mathcal{Y}$ contain multiple roles in a single contract bundle or specify overlapping sets of contracts, in which case it is impossible for the agent to participate in the set $\mathcal{Y}$ of roles, and her utility is set to a level below the utility level of her outside option. Formally, each type i has a utility $u^i:\mathcal{P}(\mathcal{R}_i)\to\mathbb{R}$ over sets of roles defined by

$$u^i(\mathcal{Y})=\begin{cases}u^i(\tau(\mathcal{Y})) & \text{if } \chi\neq\chi' \text{ and } \chi_\ell\cap\chi'_{\ell'}=\varnothing \text{ for all distinct } (\chi,\ell),(\chi',\ell')\in\mathcal{Y}\\ u^i(\varnothing)-1 & \text{otherwise}\end{cases}.$$

Given a set $\mathcal{R}\subseteq\mathcal{R}_i$ of roles, we let

$$C^i(\mathcal{R})=\underset{\mathcal{Y}\subseteq\mathcal{R}}{\operatorname{argmax}}\, u^i(\mathcal{Y})$$

denote the family of agent i's utility-maximizing choices of sets of roles in $\mathcal{R}$. By construction, if $\mathcal{Y}\in C^i(\mathcal{R})$, then the roles in $\mathcal{Y}$ must specify non-overlapping sets of contracts.

We say that a bundled outcome is individually rational if no agent wants to unilaterally drop any of the roles it participates in.

Definition 2′. *A bundled outcome $\mathcal{M}$ is* ***individually rational*** *if we have that $\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right)=0$ for each bundle-matched type $(i,\mathcal{Y})$ with $\mathcal{Y}\notin C^i(\mathcal{Y})$.*

We next introduce a concept of bundled blocks, which are blocks in which agents can sign contract bundles. We define bundled blocks similarly to blocks, but as contract bundles can involve more than two agents, bundled blocks take the shape of hypergraphs—a version of graphs that allow edges to involve more than two nodes.

Formally, a *hypergraph with n nodes* is a family ν of subsets of $\{1,2,\ldots,n\}$, each of which has at least 2 elements; the members of ν are called *hyperedges*. Given a hypergraph ν and a hyperedge $T\in\nu$, we let $T(\ell)$ denote the ℓth smallest element of T, so $T=\{T(1),\ldots,T(|T|)\}$. By convention, when we consider a contract bundle z^T for a hyperedge T, we assign the ℓth role (z^T,ℓ) in z^T to node $T(\ell)$ in the hypergraph.

Definition 3′. *Let ν be a hypergraph with n nodes. A* ***bundled block of shape*** *ν consists of*

- *a bundle-matched type $(i_j,\mathcal{Y}^j)$ for each node $1\le j\le n$, and*
- *a contract bundle $z^T\in\mathcal{X}[|T|]$ for each hyperedge $T\in\nu$ with $z^T(\ell)=i_{T(\ell)}$ for all indices $1\le\ell\le|T|$,*

for which

- *[compatibility] for each node j, the roles (z^T,ℓ) with $T(\ell)=j$ are distinct;*
- *[desirability] for each node j, writing*

$$\mathcal{Z}^j=\{(z^T,\ell)\,|\,T\in\nu \text{ and } T(\ell)=j\}$$

for the set of roles in blocking contracts assigned to the node j, we have that $\varnothing\neq\mathcal{Z}^j\subseteq\mathcal{R}_{i_j}\smallsetminus\mathcal{Y}^j$ and that $\mathcal{Z}^j\subseteq\mathcal{W}^j$ for all $\mathcal{W}^j\in C^{i_j}\left(\mathcal{Y}^j\cup\mathcal{Z}^j\right)$.

Such a bundled block ***arises*** *at a bundled outcome $\mathcal{M}$ if $\mu\left(\mathcal{M}^{i_j}_{\mathcal{Y}^j}\right)>0$ for all j.*

Here, compatibility requires that no node j participate in the same role (z^T,ℓ) multiple times (i.e., through multiple hyperedges T) in a block. Desirability requires the roles $\mathcal{Z}^j$ involved in the block not be among participating agents' existing roles $\mathcal{Y}^j$, and that the roles $\mathcal{Z}^j$ involved in the block be chosen at every selection $\mathcal{W}^j$ from participating agents' possible choices among their existing roles $\mathcal{Y}^j$ and the roles $\mathcal{Z}^j$ involved in the block.

To define tree stability, we consider an extension of the concept of acylicity to hypergraphs. We first extend the concept of paths to hypergraphs (see, e.g., Footnote 7 for a definition in the case of graphs). A *path between nodes j and k in a hypergraph ν* is a sequence $1\le j_1,\ldots,j_m\le n$ of distinct integers such that there exist distinct hyperedges $T_0,T_1,\ldots,T_m\in\nu$ with $\{j_\ell,j_{\ell+1}\}\subseteq T_\ell$ for all $1\le\ell\le m$. Here, we write $j_0=j$ and $j_{m+1}=k$. A hypergraph ν is a *generalized tree* if there exists a unique path between each pair of nodes. If each hyperedge of ν has at most two elements—i.e., if ν is actually a graph—then ν is a generalized tree if and only if it is a tree. See Figure A1 for examples of hypergraphs that are, and are

(a) The hypergraph $\{\{1,2\},\{2,3\},\{1,3,4\}\}$. (b) The hypergraph $\{\{1,2\},\{1,3,4\}\}$.

FIGURE A1

Understanding generalized trees. The hypergraph in Part (a) is not a generalized tree because it contains two paths between 1 and 3: namely the path 1,3 (using hyperedge $T_0 = \{1,3,4\}$) and the path 1,2,3 (using hyperedges $T_0' = \{1,2\}$ and $T_1' = \{2,3\}$). By contrast, the hypergraph in Part (b) is a generalized tree.

not, generalized trees. In the language of hypergraph theory, a hypergraph is a generalized tree if and only if it is Berge-acyclic and connected (Berge, 1973).[24]

We can then define tree stability and stability for bundled outcomes analogously to Definition 4, but focusing on bundled blocks and using the concept of generalized trees.

Definition 4′. *A bundled outcome is:*

- ***tree-stable*** *if it is individually rational and no bundled block of shape ν arises for any generalized tree ν.*
- ***stable*** *if it is individually rational and no bundled block of any shape arises.*

B. FURTHER RELATIONSHIPS BETWEEN STABILITY CONCEPTS

This appendix develops further relationships between stability, tree stability, the core, and additional stability concepts from the literature. Specifically, Appendix B.1 shows that stability, as well as the core, can be equivalently defined by considering (core) blocks that allow for multiple contracts between participating agents. Appendix B.2 extends Hatfield et al.'s (2013) *strong group stability* concept to large markets and uses it to derive relationships between tree stability, stability, and the core.

B.1. *Relationship of stability to Azevedo and Hatfield's (2018) definition*

In this section, we show that stability and the core (as defined in Sections 2 and 5) actually rule out (core) blocks that feature multiple contracts between pairs of agents.

We first adapt the definition of blocks (Definition 3) to allow pairs of participating agents to sign multiple contracts.

Definition B.1.1. *Let ν be a graph with n nodes. A* ***multiblock of shape*** *ν consists of*

- *a matched type (i_j, Y^j) for each node $1 \le j \le n$, and*
- *a non-empty set $Z_{j,k} = Z_{k,j} \subseteq X_{i_j,i_k}$ of contracts for each edge $\{j,k\} \in \nu$*

for which

- *[compatibility] for each node j, the sets $(Z_{j,k})_{\{j,k\} \in \nu}$ are pairwise disjoint;*

24. Bando and Hirai (2021) used the same acyclicity concept as a condition on the entire network of potential contracts to obtain an existence result for finite, multilateral matching markets.

- *[desirability] for each node j, writing*

$$Z^j = \bigcup_{\{j,k\}\in\nu} Z_{j,k},$$

we have that $\varnothing \neq Z^j \subseteq X_j \smallsetminus Y^j$ and that $Z^j \subseteq W^j$ for all $W^j \in C^{i_j}(Y^j \cup Z^j)$.

Clearly, if a block of some shape arises, then a multiblock of the same shape also arises. Indeed, one can take the set $Z_{j,k}$ of blocking contracts between the jth and kth agents in the multiblock to consist of only the contract $z_{j,k}$ between the jth and kth agents in the block. The following proposition shows that the converse also holds in large markets if the block and multiblock can be of different shapes.

Proposition B.1.1. *If a multiblock of shape ν arises at an outcome M, then a block of some shape arises at M.*

Azevedo and Hatfield's (2018) definition of stability requires that no multiblock of any shape arise (in their setting, which does not allow for indifferences between contracts). Thus, Proposition B.1.1 shows that our definition of stability coincides with Azevedo and Hatfield's (2018) definition (in their setting).

We next analogously adapt the definition of core blocks (Definition 6) to allow pairs of participating agents to sign multiple contracts.

Definition B.1.2. *A **strict** (resp., **weak**) **core multiblock of shape** ν consists of*

- *a matched type (i_j, Y^j) for each node $1 \leq j \leq n$, and*
- *a non-empty set $Z_{j,k} = Z_{k,j} \subseteq X_{i_j,i_k}$ of contracts for each edge $\{j,k\} \in \nu$*

for which

- *[compatibility] for each node j, the sets $(Z_{j,k})_{\{j,k\}\in\nu}$ are pairwise disjoint;*
- *[preferability] for each node j, writing*

$$Z^j = \bigcup_{k=1}^{n} Z_{j,k},$$

we have that $u^{i_j}(Z^j) > u^{i_j}(Y^j)$ (resp., $u^{i_j}(Z^j) \geq u^{i_j}(Y^j)$ with strict inequality for some j).

As with blocks, if a strict (resp., weak) core block of some shape arises, then a strict (resp., weak) core multiblock of the same shape also arises. The converse again holds if the core block and the core multiblock can take different shapes.

Proposition B.1.2. *If a strict (resp., weak) core multiblock of some shape arises at an outcome M, then a strict (resp., weak) core block of some shape arises at M.*

Proposition B.1.2 shows that our definition of the core coincides with the standard one.[25]

B.2. *Strong group stability*

In this appendix, we adapt Hatfield et al.'s (2013) *strong group stability* concept to large market settings with bundling, and explore the relationships between strong group stability and our other solution concepts. We use these relationships to give an equivalent definition of tree stability and prove Theorems 3 and 4.

To define strong group stability in our setting, we weaken the desirability condition in the definition of bundled blocks (Definition 3′) to a condition that we call *group desirability*. The difference between desirability and group desirability is that the latter condition only requires that agents could increase their utility by choosing all of their roles in the blocking contract bundles (possibly in addition to some existing roles), rather than that agents would necessarily select all of these roles in an optimal choice.

25. See, e.g., Azevedo and Hatfield (2018) for a definition in a large-market setting without indifferences.

Definition B.2.1. *Let ν be a hypergraph with n nodes. A **bundled group block of shape** ν consists of*

- *a bundle-matched type $(i_j, \mathcal{Y}^j)$ for each node $1 \leq j \leq n$, and*
- *a contract bundle $z^T \in \mathcal{X}[|T|]$ for each hyperedge $T \in \nu$ with $z^T(\ell) = i_{T(\ell)}$ for all indices $1 \leq \ell \leq |T|$,*

for which

- *[compatibility] for each node j, the roles (z^T, ℓ) with $T(\ell) = j$ are distinct;*
- *[group desirability] for each node j, writing*

$$\mathcal{Z}^j = \{(z^T, \ell) \mid T \in \nu \text{ and } T(\ell) = j\}$$

for the set of roles in blocking contracts assigned to the node j, we have that $\varnothing \neq \mathcal{Z}^j \subseteq \mathcal{R}_{i_j}$, and there exists a set $\mathcal{W}^j \subseteq \mathcal{Y}^j \cup \mathcal{Z}^j$ with $\mathcal{Z}^j \subseteq \mathcal{W}^j$ and $u^{i_j}(\mathcal{W}^j) > u^{i_j}(\mathcal{Y}^j)$.

Here, group desirability requires that there be a set $\mathcal{W}^j$ (consisting of some of the existing roles and some new roles) that delivers higher utility than the existing roles, while the desirability condition in Definition 3′ requires that each participating agent would necessarily choose all of the new roles (when given access to both the new and the existing roles).[26] Strong group stability refines stability by considering bundled group blocks instead of bundled blocks.

Definition B.2.2. *A bundled outcome is* strongly group stable *if it is individually rational and no bundled group block of any shape arises.*

We next formally show that desirability implies group desirability in our model, as Hatfield et al. (2013) pointed out in finite markets.

Lemma B.2.1. *Each bundled block of shape ν is a bundled group block of shape ν. In particular, every strongly group stable bundled outcome is stable.*

Hatfield et al. (2013) showed that the converse of Lemma B.2.1 is not generally true, even in finite markets (see Section 4.2 of Hatfield et al. (2013)). However, a version of this converse does turn out to hold when one restricts attention to (bundled group) blocks of the shapes of generalized trees.

Proposition B.2.1. *If a bundled group block of the shape of a generalized tree arises at a bundled outcome $\mathcal{M}$, then $\mathcal{M}$ is not tree-stable.*

Lemma B.2.1 and Proposition B.2.1 show that tree stability can be defined equivalently using group desirability instead of desirability. This equivalence result in fact holds in finite markets as well, and is new even in that case.

Strong group stability also provides a refinement of the core as long as each contract bundle consisting of a single contract is feasible. Intuitively, group desirability refines the preferability condition of Definition 6 by allowing agents to maintain some existing contracts. Hatfield et al. (2013) described this relationship in the case of finite markets with transferable utility.

Lemma B.2.2. *If each contract bundle that consists of a single contract is feasible, then each strongly group stable bundled outcome is equivalent up to bundling to a weak core outcome.*

As discussed in Section 5.4, strong group stability also lets us unify Theorems 3 and 4.

Theorem B.2.1. *For each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then:*

26. Unlike in finite markets, strong group stability coincides with a version of group stability (Konishi and Ünver (2006); called strong setwise stability by Klaus and Walzl (2009)), as Vocke (2022) shows.

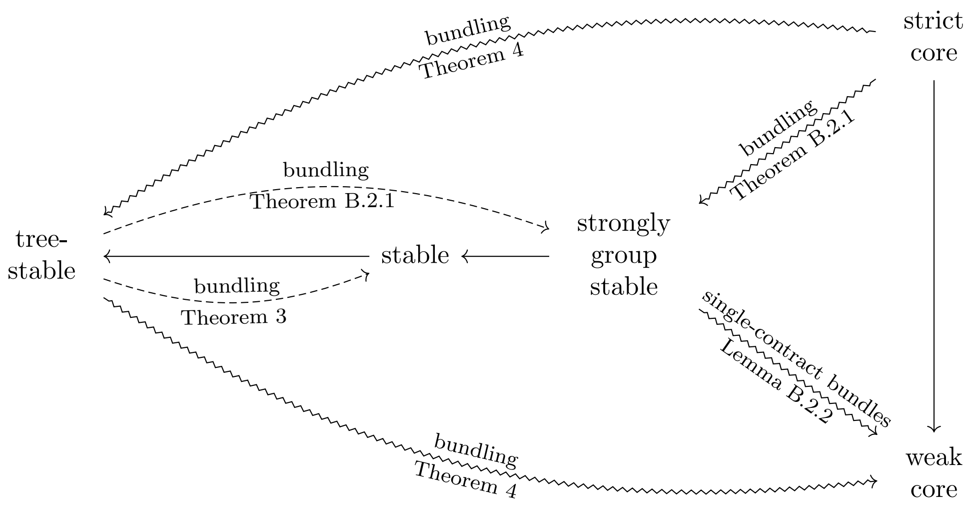


FIGURE A2

Summary of the relationships between solution concepts. The regular arrows represent obvious relationships between solution concepts. The dashed arrows represent relationships that rely on all bundles among small groups of agents being feasible. The squiggly arrows represent relationships between solution concepts modulo equivalence up to bundling, which also rely on all bundles among small groups of agents, or at least all single-contract bundles, being feasible.

(a) Every tree-stable bundled outcome is strongly group stable.

(b) Each strict core outcome is equivalent up to bundling to a strongly group stable bundled outcome.

In light of Lemmata B.2.1 and B.2.2, Theorem B.2.1(a) unifies Theorems 3 and 4(a). Theorem B.2.1(b) shows that Theorem 4(b) extends to a result about strong group stability as well. Note that Theorem B.2.1 actually holds in finite markets as well, and is a new result even in that context.

Figure A2 summarizes the relationships between the solution concepts.

C. PROOFS

C.1. *Proof of Theorem 2*

Define a set

$$D=\left\{d\in\bigtimes_{i\in I}[0,\theta^i]^{\mathcal{P}(X_i)}\;\middle|\;\sum_{Y\in\mathcal{P}(X_i)}d^i_Y=\theta^i \text{ for all } i\right\},$$

which represents the set of possible distributions of matched types. Here, for $d\in D$, we write $d=(d^i)_{i\in I}$, where $d^i\in[0,\theta^i]^{\mathcal{P}(X_i)}$. For each contract $x\in X_{i,j}$, let $\lambda_x=\max\{\theta^i,\theta^j\}$. Define a set

$$O=\left\{o\in\bigtimes_{i\in I}\mathbb{R}^{X_i}\;\middle|\;o^i_x\le\lambda_x \text{ for all } i\in I \text{ and } x\in X_i\right\},$$

which denotes the set of possible "offer vectors" consisting of masses of contracts offered to each type. Here, for $o\in O$, we write $o=(o^i)_{i\in I}$, where $o^i\in\mathbb{R}^{X_i}$. Define $\overline{o}\in O$ by $\overline{o}^i_x=\lambda_x$ for $i\in I$ and $x\in X_i$.

There is a natural aggregation operator $\pi:D\to O$ defined by

$$\pi(d)^i_x=\sum_{Y\in\mathcal{P}(X_i)|x\in Y}d^i_Y$$

for each type i and contract $x \in X_i$. We define an aggregate choice correspondence $h : O \rightrightarrows D$ by

$$h(o) = \underset{d \in D \mid \pi(d) \leq o}{\operatorname{argmax}} \sum_{i \in I} \sum_{Y \in \mathcal{P}(X_i)} d^i_Y u^i(Y).$$

That is given an vector $o \in O$ of offers of masses of contracts to each type, the operator h returns (in the form of a distribution of matched types) the set of all possible utility-maximizing allocations of the masses to agents.[27]

Define an operator $\Upsilon : O \to O$ by $\Upsilon(o)^i_x = o^j_x$ for all $x \in X_{i,j}$, which exchanges the masses offered between the counterparty types of each contract. We then define a generalized Gale–Shapley operator $\Phi : O \rightrightarrows O$ by

$$\Phi(o) = \{\Upsilon(\overline{o} - o + \pi(d)) \mid d \in h(o)\}.$$

For intuition, d represents a selection from aggregate demand for contracts from o, so $o - \pi(d)$ represents a selection from (aggregate) rejected contracts. Thus, $\Phi(o)$ takes the complement $\overline{o} - (o - \pi(d))$ of a selection $o - \pi(d)$ from (aggregate) rejected contracts and (applying Υ) offers masses to counterparties—so Φ adapts Che et al.'s (2021) Gale–Shapley operator to a continuum network setting and Hatfield and Kominers's (2012) operator to continuum setting with indifferences.[28] We have that $\Phi(o) \subseteq O$ because $0 \leq \pi(d) \leq o$ for all $d \in h(o)$ by construction.

As h is the solution of a linear optimization problem with compact, convex domain and affine constraints, the correspondence h is nonempty, compact, convex-valued. It follows from Berge's Theorem of the Maximum that h is upper hemicontinuous. As π is linear, the correspondence Φ is upper hemicontinuous and nonempty, compact, convex-valued as well. Hence, Kakutani's Fixed Point Theorem guarantees that Φ has a fixed point.

Let $o \in O$ be a fixed point—i.e., such that $o \in \Phi(o)$. By the definition of Φ, there exists $d \in h(o)$ such that $o = \Upsilon(\overline{o} - o + \pi(d))$. Noting that $\Upsilon \circ \Upsilon$ is the identity, applying Υ yields that $\Upsilon(o) = \overline{o} - o + \pi(d)$. Rearranging terms, we have that $\pi(d) = o + \Upsilon(o) - \overline{o}$. Hence, for all types i,j and contracts $x \in X_{i,j}$, we have that

$$\pi(d)^i_x = o^i_x + \Upsilon(o)^i_x - \overline{o}^i_x = \Upsilon(o)^j_x + o^j_x - \overline{o}^j_x = \pi(d)^j_x, \tag{C.1}$$

where the second equality holds as $(\Upsilon(o)^i_x, \Upsilon(o)^j_x) = (o^j_x, o^i_x)$ and $\overline{o}^i_x = \overline{o}^j_x$ by construction.

We construct an outcome from d, which we then show is tree-stable. Specifically, we choose an arbitrary ordering $\mathcal{P}(X_i) = \{Y^1, \ldots, Y^{|\mathcal{P}(X_i)|}\}$ of $\mathcal{P}(X_i)$ for each i and define[29]

$$M^i_{Y^\ell} = \begin{cases} \{i\} \times \left[\sum_{k<\ell} d^i_{Y^k}, \sum_{k \leq \ell} d^i_{Y^k}\right) & \text{if } \ell < |\mathcal{P}(X_i)| \\ \{i\} \times \left[\sum_{k<|\mathcal{P}(X_i)|} d^i_{Y^k}, \sum_{k \leq |\mathcal{P}(X_i)|} d^i_{Y^k}\right] & \text{if } \ell = |\mathcal{P}(X_i)| \end{cases}.$$

The feasibility condition from Definition 1 is clearly satisfied and (C.1) ensures that the reciprocity condition from Definition 1 is satisfied. Hence, $M = (M^i_Y)_{i \in I, Y \in \mathcal{P}(X_i)}$ is an outcome. By construction, a matched type (i,Y) arises at M if and only if $d^i_Y > 0$.

We next show that M is individually rational. Let (i,Y) be a matched type with $Y \notin C^i(Y)$. Let $Z \in C^i(Y)$ be arbitrary. If $d^i_Y > 0$, then letting $\hat{d} \in D$ be defined by $\hat{d}^j = d^j$ for $j \neq i$ and $\hat{d}^i = (d^i_{\mathcal{P}(X_i) \setminus \{Y,Z\}}, 0_Y, (d^i_Y + d^i_Z)_Z)$, we have that $\pi(\hat{d}) \leq \pi(d) \leq o$ and that

$$\sum_{j \in I} \sum_{Y' \in \mathcal{P}(X_j)} d^j_{Y'} u^j(Y') < \sum_{j \in I} \sum_{Y' \in \mathcal{P}(X_j)} \hat{d}^j_{Y'} u^j(Y').$$

Hence, by the definition of h, since $d \in h(o)$, we must have that $d^i_Y = 0$ for all matched types (i,Y) with $Y \notin C^i(Y)$. It follows that the outcome M is individual rational.

To show that no block of the shape of a tree can arise at M, we use the following claim.

27. This contruction adapts a related construction of Greinecker and Kah (2021) from a one-to-one setting, which itself adapted a construction of Jagadeesan (2017).

28. This approach builds on the operators constructed by Fleiner (2003) and Hatfield and Milgrom (2005). We cannot use Azevedo and Hatfield's (2018) operator (which is based on the operators of Adachi (2000), Echenique and Oviedo (2004), and Ostrovsky (2008)) due to the presence of indifferences in our setting.

29. The construction of the bundled outcome is arbitrary; what is relevant is the masses of agents of each bundle-matched type.

Claim C.1.1. *Let $o \in O$ be such that $o \in \Phi(o)$, and let $d \in h(o)$ be such that $\Upsilon(o) = \overline{o} - o + \pi(d)$. Let (i,Y) be a matched type and let $\{x^1,\ldots,x^n\} \subseteq X_i \smallsetminus Y$ be such that there exists $Z \subseteq Y \cup \{x^1,\ldots,x^n\}$ with $u^i(Z) > u^i(Y)$. If $d^i_Y > 0$, then there exists $1 \le k \le n$ such that—writing $x^k \in X_{i,i'_k}$—we have that $o^i_{x^k} < o^{i'_k}_{x^k}$.*

Proof. Letting $\hat{d} \in D$ be defined by $\hat{d}^{i'} = d^{i'}$ for $i' \neq i$ and $\hat{d}^i = (d^i_{\mathcal{P}(X_i) \smallsetminus \{Y,Z\}}, d^i_Y - \delta, d^i_Z + \delta)$ for any $\delta > 0$, we have that

$$\sum_{i' \in I} \sum_{Y' \in \mathcal{P}(X_{i'})} d^{i'}_{Y'} u^{i'}(Y') < \sum_{i' \in I} \sum_{Y' \in \mathcal{P}(X'_i)} \hat{d}^{i'}_{Y'} u^{i'}(Y').$$

Hence, as $d \in h(o)$, we must have that $\pi(\hat{d}) \not\le o$. In particular, there must exist $1 \le k \le n$ such that $\pi(d)^i_{x^k} = o^i_{x^k}$. By (C.1), it follows that $o^{i'_k}_{x^k} = \overline{o}^i_{x^k}$. As $d^i_Y > 0$ and $x^k \notin Y$, we have that $\pi(d)^i_{x^k} < \theta^i$ and it follows that

$$o^i_{x^k} = \pi(d)^i_{x^k} < \theta^i \le \lambda_{x^k} = \overline{o}^i_{x^k} = o^{i'_k}_{x^k},$$

as desired. ‖

Suppose for sake of deriving a contradiction that there is a tree ν and a block of shape ν that arises at M. Let the constituent matched types of the block be $(i_1,Y^1),\ldots,(i_n,Y^n)$ and the constituent contracts be $(x_{j,k})_{\{j,k\} \in \nu}$. Construct a directed graph with node set $\{1,\ldots,n\}$ in which there is an edge from j to k if and only if $\{j,k\} \in \nu$ and $o^{i_j}_{x_{j,k}} < o^{i_k}_{x_{j,k}}$. By Claim C.1.1, each node must have out-degree at least 1. It follows that this directed graph must have a directed cycle, which contradicts the hypothesis that ν is a tree.

Hence, we can conclude that no block of the shape of a tree arises at M. It follows that M is tree-stable—as desired.

C.2. *Proof of Theorem 2′*

The proof follows a similar approach to the proof of Theorem 2. The key technical challenge relative that argument is that the setting with bundling introduces the possibility of multilateral contracting via contract bundles. Conceptually, we deal with this issue by considering an auxiliary two-sided matching market in which agents match with contract bundles; the preferences of a contract bundle χ in this market are such that they only find it feasible to match with the agents taking all roles in χ, or no agents at all. Technically, rather than explicitly constructing the auxiliary market, we instead work directly with the Gale–Shapley operator (as constructed in the proof of Theorem 2) for the auxiliary market. We use the assumption that $\mathcal{X}[n] = \varnothing$ for all sufficiently large n to ensure that the sets of contract bundles and of roles are finite, and therefore that the Gale-Shapley operator is defined on a finite-dimensional space.

Define a set

$$D = \left\{ d \in \bigtimes_{i \in I} [0,\theta^i]^{\mathcal{P}(\mathcal{R}_i)} \;\middle|\; \sum_{\mathcal{Y} \in \mathcal{P}(\mathcal{R}_i)} d^i_{\mathcal{Y}} = \theta^i \text{ for all } i \right\},$$

which represents the set of possible distributions of bundle-matched types. Here, for $d \in D$, we write $d = (d^i)_{i \in I}$, where $d^i \in [0,\theta^i]^{\mathcal{P}(\mathcal{R}_i)}$. For each n and each contract bundle $\chi \in \mathcal{X}[n]$, let

$$\lambda_\chi = \max_{1 \le \ell \le n} \{\theta^{\chi(\ell)}\}.$$

Define a set

$$O = \left\{ o \in \mathbb{R}^{\bigcup_{i \in I} \mathcal{R}_i}_{\ge 0} \;\middle|\; o_{(\chi,\ell)} \le \lambda_\chi \text{ for all } 1 \le \ell \le n \text{ and } \chi \in \mathcal{X}[n] \right\},$$

which denotes the set of possible "offer vectors" consisting of masses of roles offered to each type. Define $\overline{o} \in O$ by $\overline{o}_{(\chi,\ell)} = \lambda_\chi$ for all $1 \le \ell \le n$ and $\chi \in \mathcal{X}[n]$.

There is a natural aggregation operator $\pi : D \to O$ defined by

$$\pi(d)_{(\chi,\ell)} = \sum_{\mathcal{Y} \subseteq \mathcal{P}(\mathcal{R}_{\chi(\ell)}) \mid (\chi,\ell) \in \mathcal{Y}} d^{\chi(\ell)}_{\mathcal{Y}}$$

for each role $(\chi,\ell) \in \bigcup_{i \in I} \mathcal{R}_i$. We define an aggregate choice correspondence $h : O \rightrightarrows D$ by

$$h(o) = \operatorname*{argmax}_{d \in D \mid \pi(d) \le o} \sum_{i \in I} \sum_{\mathcal{Y} \in \mathcal{P}(\mathcal{R}_i)} d^i_{\mathcal{Y}} u^i(\mathcal{Y}).$$

That is, given a vector $o \in O$ of offers of masses of roles to each type, the operator h returns (in the form of a distribution of bundle-matched types) the set of all allocations of the masses to agents that maximize total utility.

Define an operator $\Psi: O \to O$ by

$$\Psi(o)_{(\chi,\ell)} = \min_{1\le k\le n} o_{(\chi,k)}$$

for all $1\le \ell \le n$ and $\chi \in X[n]$. Note that $0 \le \Psi(o) \le o$ holds for all $o\in O$ by construction. We then define a generalized Gale–Shapley operator $\Phi: O\times O \rightrightarrows O\times O$ by

$$\Phi(o,o') = \Phi_1(o') \times \{\Phi_2(o)\},$$

where

$$\Phi_1(o') = \{\bar{o} - o' + \pi(d) \mid d \in h(o')\}$$

$$\Phi_2(o) = \bar{o} - o + \Psi(o).$$

For intuition, it is helpful to consider the auxiliary two-sided market described at the start of the proof. In that market, o (resp. o') represents the mass of contracts offered to the contract bundle side (resp. agent side). Furthermore, since either all roles or no roles in a contract bundle must be taken, $\Psi(o)$ represents the aggregate choice of the contract bundle side from o in the auxiliary market. Hence, $o - \Psi(o)$ represents a selection from (aggregate) rejection from o by the contract bundle slide, and the operator Φ takes the complement $\Phi_2(o) = \bar{o} - (o - \Psi(o))$, and sets it as the offer o' to the agent side. Dually, d represents a selection from aggregate demand from o' by the agent side, so $o' - \pi(d)$ represents a selection from (aggregate) rejection by the agent side. Hence, the operator Φ takes the complement $\bar{o} - (o' - \pi(d)) \in \Phi_1(o')$ of a selection $o' - \pi(d)$ from aggregate rejection by the agent side, and sets it as the offer o to the contract bundle side. Thus, Φ adapts Che et al.'s (2021) Gale–Shapley operator to the auxiliary two-sided market.

As $0 \le \pi(d) \le o$ for all $d \in h(o')$, we have that $\Phi_1(o') \subseteq O$ for all $o' \in O$; similarly, as $0 \le \Psi(o) \le o$, we have that $\Phi_2(o) \in O$ for all $o \in O$.

Since $X[n] = \varnothing$ for all sufficiently large n, the spaces D and O are finite-dimensional. Hence, h is the solution of a linear optimization problem with compact, convex domain and affine constraints; therefore, the correspondence h is nonempty, compact, convex-valued. It follows from Berge's Theorem of the Maximum that h is upper hemicontinuous. As π is linear, the correspondence Φ_1 is upper hemicontinuous and nonempty, compact, convex-valued as well. The function Φ_2 is clearly continuous. Hence, Kakutani's Fixed Point Theorem guarantees that the correspondence Φ has a fixed point.

Let $(o,o') \in O\times O$ be a fixed point—that is, a pair (o,o') with $(o,o') \in \Phi(o,o')$. By the definition of Φ, there exists $d \in h(o')$ such that $o = \bar{o} - o' + \pi(d)$, and we have that $o' = \Phi_2(o) = \bar{o} - o + \Psi(o)$. Thus, we have that $\pi(d) = o + o' - \bar{o} = \Psi(o)$. Hence, for all $1 \le k,\ell \le n$ and $\chi \in X[n]$, we have that

$$\pi(d)_{(\chi,\ell)} = \Psi(o)_{(\chi,\ell)} = \Psi(o)_{(\chi,k)} = \pi(d)_{(\chi,k)}, \tag{C.1}$$

where the second equality follows from the definition of Ψ.

We construct a bundled outcome from d, which we then show to be tree-stable. Specifically, we choose an arbitrary ordering $\mathcal{P}(\mathcal{R}_i) = \{\mathcal{Y}^1, \ldots, \mathcal{Y}^{|\mathcal{P}(\mathcal{R}_i)|}\}$ of $\mathcal{P}(\mathcal{R}_i)$ for each i and define

$$\mathcal{M}^i_{\mathcal{Y}^\ell} = \begin{cases} \{i\} \times \left[\sum_{k<\ell} d^i_{\mathcal{Y}^k}, \sum_{k\le \ell} d^i_{\mathcal{Y}^k}\right) & \text{if } \ell < |\mathcal{P}(\mathcal{R}_i)| \\ \{i\} \times \left[\sum_{k<|\mathcal{P}(\mathcal{R}_i)|} d^i_{\mathcal{Y}^k}, \sum_{k\le |\mathcal{P}(\mathcal{R}_i)|} d^i_{\mathcal{Y}^k}\right] & \text{if } \ell = |\mathcal{P}(\mathcal{R}_i)| \end{cases}.$$

The feasibility condition from Definition 1$'$ is clearly satisfied and (C.1) ensures that the reciprocity condition from Definition 1$'$ is satisfied. Hence, $\mathcal{M} = (\mathcal{M}^i_{\mathcal{Y}})_{i,\mathcal{Y}}$ is a bundled outcome. By construction, a bundle-matched type $(i,\mathcal{Y})$ arises at $\mathcal{M}$ if and only if $d^i_{\mathcal{Y}} > 0$.

We next show that $\mathcal{M}$ is individually rational. Let $(i,\mathcal{Y})$ be a bundle-matched type with $\mathcal{Y} \notin C^i(\mathcal{Y})$. Let $\mathcal{Z} \in C^i(\mathcal{Y})$ be arbitrary. If $d^i_{\mathcal{Y}} > 0$, then defining $\hat{d} \in D$ by $\hat{d}^j = d^j$ for $j \ne i$ and $\hat{d}^i = (d^i_{\mathcal{P}(\mathcal{R}_i)\smallsetminus\{\mathcal{Y},\mathcal{Z}\}}, 0_{\mathcal{Y}}, (d^i_{\mathcal{Y}} + d^i_{\mathcal{Z}})_{\mathcal{Z}})$, we have that $\pi(\hat{d}) \le \pi(d) \le o'$ and that

$$\sum_{j\in I} \sum_{\mathcal{Y}'\in\mathcal{P}(\mathcal{R}_j)} d^j_{\mathcal{Y}'} u^j(\mathcal{Y}') < \sum_{j\in I} \sum_{\mathcal{Y}'\in\mathcal{P}(\mathcal{R}_j)} \hat{d}^j_{\mathcal{Y}'} u^j(\mathcal{Y}').$$

Hence, by the definition of h, since $d \in h(o')$, we must have that $d^i_{\mathcal{Y}} = 0$ for all bundle-matched types $(i,\mathcal{Y})$ with $\mathcal{Y} \notin C^i(\mathcal{Y})$. It follows that $\mathcal{M}$ is individually rational.

To show that no bundled block of the shape of a generalized tree can arise at $\mathcal{M}$, we use the following claim, which is an analogue of Claim C.1.1.

Claim C.2.1. *Let $(o,o')\in O\times O$ be such that $(o,o')\in\Phi(o,o')$, and let $d\in h(o')$ be such that $\pi(d)=\Psi(o)$. Let $(i,\mathcal{Y})$ be a bundle-matched type and let $\{r^1,\ldots,r^n\}\subseteq\mathcal{R}_i\setminus\mathcal{Y}$ be such that there exists $\mathcal{Z}\subseteq\mathcal{Y}\cup\{r^1,\ldots,r^n\}$ with $u^i(\mathcal{Z})>u^i(\mathcal{Y})$. If $d^i_{\mathcal{Y}}>0$, then there exists $1\le k\le n$ such that—writing $r^k=(\chi^k,\ell_k)$ with $\chi^k\in\mathcal{X}[m]$—we have that*

$$o'_{r^k}<\max_{1\le\ell\le m}o'_{(\chi^k,\ell)}.$$

Proof. Note that since $(o,o')\in\Phi(o,o')$, we have that $o'=\Phi_2(o)$; since $\Phi_2(o)=\overline{o}-o+\Psi(o)$ (by the definition of Φ_2) and $\pi(d)=\Psi(o)$, we then have that

$$o'=\Phi_2(o)=\overline{o}-o+\Psi(o)=\overline{o}-o+\pi(d). \tag{C.2}$$

We first construct the role r^k. Define $\hat{d}\in D$ by $\hat{d}^j=d^j$ for $j\neq i$ and

$$\hat{d}^i=(d^i_{\mathcal{P}(\mathcal{R}_i)\setminus\{\mathcal{Y},\mathcal{Z}\}},d^i_{\mathcal{Y}}-\delta,d^i_{\mathcal{Z}}+\delta),$$

where $0<\delta<d^i_{\mathcal{Y}}$. By construction, since $u^i(\mathcal{Z})>u^i(\mathcal{Y})$, we have that

$$\sum_{j\in I}\sum_{\mathcal{Y}'\in\mathcal{P}(\mathcal{R}_j)}d^j_{\mathcal{Y}'}u^j(\mathcal{Y}')<\sum_{j\in I}\sum_{\mathcal{Y}'\in\mathcal{P}(\mathcal{R}_j)}\hat{d}^j_{\mathcal{Y}'}u^j(\mathcal{Y}').$$

Hence, since $d\in h(o')$, we must have that $\pi(\hat{d})\not\le o'$ for all $0<\delta<d^i_{\mathcal{Y}}$. It follows that there exists $1\le k\le n$ such that $\pi(d)_{r^k}=o'_{r^k}$.

We now prove the assertion of the claim. Note that

$$o'_{r^k}=\pi(d)_{r^k}=\pi(d)_{(\chi^k,\ell)} \tag{C.3}$$

for all $1\le\ell\le m$, where the second equality comes from (C.1). Since $d\in h(o')$, for all $1\le\ell\le m$, we have that $\pi(d)_{(\chi^k,\ell)}\le o'_{(\chi^k,\ell)}$; we thus have that $o'_{r^k}\le o'_{(\chi^k,\ell)}$.

We need to show that $o'_{r^k}<o'_{(\chi^k,\ell)}$ for some $1\le\ell\le m$. Suppose for sake of deriving a contradiction that $o'_{r^k}=o'_{(\chi^k,\ell)}$ for all $1\le\ell\le m$. In this case, it follows from (C.2) and (C.3) that

$$\pi(d)_{(\chi^k,\ell)}=o'_{r^k}=o'_{(\chi^k,\ell)}=\overline{o}_{(\chi^k,\ell)}-o_{(\chi^k,\ell)}+\pi(d)_{(\chi^k,\ell)}$$

for all $1\le\ell\le m$. Hence, we must have that $o_{(\chi^k,\ell)}=\overline{o}_{(\chi^k,\ell)}$ for all $1\le\ell\le m$. As $\overline{o}_{(\chi^k,\ell)}=\lambda_{\chi^k}$ for all $1\le\ell\le m$ (by the definition of $\overline{o}$), the definition of $\Psi(o)$ implies that $\Psi(o)_{r^k}=\lambda_{\chi^k}$. Since $\pi(d)=\Psi(o)$, we must have that $\pi(d)_{r^k}=\lambda_{\chi^k}$. But since $d^i_{\mathcal{Y}}>0$ and $r^k\notin\mathcal{Y}$, we have that $\pi(d)_{r^k}<\theta^i\le\lambda_{\chi^k}$—a contradiction. ‖

Suppose for sake of deriving a contradiction that there is a generalized tree ν with n nodes and a bundled block of shape ν that arises at $\mathcal{M}$. Let the bundled block consist of bundle-matched types $(i_j,\mathcal{Y}^j)$ for $1\le j\le n$ and contract bundles $\chi^T\in\mathcal{X}[|T|]$ for $T\in\nu$. We now construct a graph ν' from ν in the following way: fix an arbitrary ordering $\nu=\{T_1,\ldots T_{|\nu|}\}$, let the nodes of ν' be given by $\{1,\ldots,n,n+1,\ldots,n+|\nu|\}$. There is an edge from j to k in ν' if and only if exactly one of j and k is less than or equal to n and, supposing that $j\le n$, we have that $j\in T_{k-n}$. Since the hypergraph ν is a generalized tree, the graph ν' is a tree by construction.

We now direct the edges of ν' as follows. Consider an edge $\{j,k\}\in\nu'$, where $j\le n$. We direct the edge from j to k if

$$o'_{\left(\chi^{T_{k-n}},T^{-1}_{k-n}(j)\right)}<\max_{1\le\ell\le|T_{k-n}|}o'_{\left(\chi^{T_{k-n}},\ell\right)}$$

and from k to j otherwise. Here, $T^{-1}_{k-n}(j)$ is a well-defined element of $\{1,2,\ldots,|T_{k-n}|\}$ since $j\in T_{k-n}$.

We next show that each node in ν' has out-degree at least 1. This property holds for nodes $k>n$ by construction: letting

$$\ell^k\in\operatorname*{argmax}_{1\le\ell\le|T_{k-n}|}o'_{\left(\chi^{T_{k-n}},\ell\right)},$$

the edge between k and $T_{k-n}(\ell^k)$ is directed from k to $T_{k-n}(\ell^k)$. For nodes $j\le n$, considering the set $\mathcal{Z}^j$ of new roles defined in Definition 3′, Claim C.2.1 implies that there exists a role $r\in\mathcal{Z}^j$ such that, writing $r=(\chi^{T_{k-n}},\ell)$, the edge between j and k is directed from j to k. It follows that ν' must have a directed cycle, which contradicts the fact that the undirected version of ν' forms a tree.

Hence, we can conclude that no block of the shape of a generalized tree arises at $\mathcal{M}$. It follows that $\mathcal{M}$ is tree-stable—as desired.

C.3. *Proof of Proposition 1*

Every stable outcome is clearly tree-stable.

To prove the inverse, consider an outcome M that is not stable. If M is not individually rational, then clearly M is not tree-stable. Hence, we can assume that a block of the shape of some graph ν with n nodes arises in M—say defined by the matched types $(i_1,Y^1),\ldots,(i_n,Y^n)$ and the contracts $z_{j,k}$ for $\{j,k\}\in\nu$. We construct a tree ν' from ν such that a block of shape ν' arises at M. Intuitively, we consider the blocking contracts signed by one buyer involved in the original block to construct a star-shaped network centered around a buyer; substitutability for sellers ensures that this tree is a block.

More formally, since the market is two-sided with buyer types B and seller types S, at least one of the types $i_1,\ldots,i_n$ must be in B. Without loss of generality, suppose that $i_1\in B$, and that $\{1,k\}\in\nu$ if and only if $2\leq k\leq m$. Consider the graph ν' with m nodes defined by $\nu'=\{\{1,k\}\,|\,2\leq k\leq m\}$. This graph is clearly a tree. We claim that the matched types $(i_1,Y^1),\ldots,(i_m,Y^m)$ and the contracts $(z_{1,k})_{2\leq k\leq m}$ comprise a block. The compatibility condition follows from the compatibility condition for the original block of shape ν. To show the desirability condition, for $1\leq j\leq m$, let

$$Z'^{,j}=\{z'_{j,k}\,|\,\{j,k\}\in\nu'\}$$

denote the set of contracts in which the jth agent participates in the candidate block of shape ν'. We divide into cases to show that for all $1\leq j\leq m$, we have that $W'^{,j}\supseteq Z'^{,j}$ for all $W'^{,j}\in C^{i_j}(Y^j\cup Z'^{,j})$.

Case 1: $j=1$. Letting Z^1 be as in Definition 3 for the original block of shape ν, we have that $Z'^{,1}=Z^1$ by construction. Hence, the desirability condition from Definition 3 for the original block of shape ν implies that $W'^{,1}\supseteq Z'^{,1}$ for all $W'^{,1}\in C^{i_1}(Y^1\cup Z'^{,1})$.

Case 2: $j>1$. The two-sidedness of the market and the hypothesis that $i_1\in B$ together imply that $i_j\in S$. Hence, type i_j's preferences are substitutable by hypothesis.

Letting Z^j be as in Definition 3 for the original block of shape ν, we have that $Z'^{,j}\subseteq Z^j$ by construction. The desirability condition from Definition 3 for the original block of shape ν guarantees that $W^j\supseteq Z^j$ for all $W^j\in C^{i_j}(Y^j\cup Z^j)$. Substitutability then implies that $W'^{,j}\supseteq Z'^{,j}$ for all $W^j\in C^{i_j}(Y^j\cup Z'^{,j})$.

The cases exhaust all possibilities, and we can conclude that we have defined a block of shape ν'. The block arises at M because the original block of shape ν arises at M.

C.4. *Proof of Theorem 3*

Theorem B.2.1(a) guarantees that for each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then every tree-stable bundled outcome is strongly group stable. As strongly group stable bundled outcomes are stable (Lemma B.2.1), the theorem follows.

C.5. *Proof of Theorem 4*

Theorem B.2.1 guarantees that for each set X of contracts, there exists a constant N such that if all contract bundles involving at most N agents are feasible, then every tree-stable bundled outcome is strongly group stable and every strict core outcome is equivalent up to bundling to a strongly group stable bundled outcome. As each strongly group stable bundled outcome equivalent up to bundling to a weak core outcome (Lemma B.2.2) and stable (Lemma B.2.1), the theorem follows.

C.6. *Proof of Propositions B.1.1 and B.1.2*

Consider a multiblock (resp., strict core multiblock, weak core multiblock) of the shape of some graph ν with n nodes that arises at M—say defined by the matched types $(i_1,Y^1),\ldots,(i_n,Y^n)$ and the contract sets $Z_{j,k}$ for $\{j,k\}\in\nu$. We construct a graph ν' from ν such that some block of shape ν' arises at M. Intuitively, we define ν' to contain enough copies of the nodes in ν to make each edge of ν' associated to only one contract. Specifically, we let $q=\max_{\{j,k\}\in\nu}|Z_{j,k}|$ and construct ν' by taking q copies of each node of ν.

Formally, we write

$$\{1,2,\dots,n\}\times\{1,2,\dots,q\}=\{(a_1,b_1),\dots,(a_{qn},b_{qn})\}.$$

Furthermore, for each $\{j,k\}\in\nu$, we write

$$Z_{j,k}=\left\{z^1_{j,k},\dots,z^{|Z_{j,k}|}_{j,k}\right\},$$

where $z^p_{j,k}=z^p_{k,j}$. Define a graph ν' with qn nodes by, when $a_j\le a_k$, letting $\{j,k\}\in\nu'$ if $a_j<a_k$ and $b_k-b_j\equiv p \pmod q$ for some p with $1\le p\le |Z_{a_j,a_k}|$.

For each $1\le j\le qn$, we define a matched type $(i'_j,Y'^{,j})=(i_{a_j},Y^{a_j})$. For each edge $\{j,k\}\in\nu'$, supposing that $a_j<a_k$ and letting $1\le p\le|Z_{a_j,a_k}|$ be such that $b_k-b_j\equiv p \pmod q$, we define a contract $z'_{j,k}=z^p_{a_j,a_k}$. We claim that these matched types and contracts define a block (resp., strict core block, weak core block) of shape ν'. To see this, note that, for each $1\le j\le qn$, the set of blocking contracts assigned to j' is

$$Z'^{,j}=\left\{z'_{j,k}\,\middle|\,\{j,k\}\in\nu'\right\}=\bigcup_{a_k\,|\,\{a_j,a_k\}\in\nu} Z_{a_j,a_k}=Z^{a_j}.$$

As the matched types (i_j,Y^j) arise for $1\le j\le n$, this block (resp., strict core block, weak core block) arises at M.

C.7. *Proof of Lemma B.2.1*

Suppose that ν is a hypergraph with n nodes and that the bundle-matched types $(i_j,\mathcal{Y}^j)_{1\le j\le n}$ and contract bundles $(z^T)_{T\in\nu}$ comprise a bundled block. Consider $\mathcal{W}^j\in\mathcal{C}^{i_j}(\mathcal{Y}^j\cup\mathcal{Z}^j)$. The desirability condition in Definition 3′ implies that $\mathcal{W}^j\supseteq\mathcal{Z}^j$. It also implies that $\mathcal{Y}^j\notin\mathcal{C}^{i_j}(\mathcal{Y}^j\cup\mathcal{Z}^j)$; it follows that $u^{i_j}(\mathcal{W}^j)>u^{i_j}(\mathcal{Y}^j)$. Thus, the group desirability condition in Definition B.2.1 must hold, so the bundled block is in fact a bundled group block.

C.8. *Proof of Proposition B.2.1*

If $\mathcal{M}$ is not individually rational, then $\mathcal{M}$ is not tree-stable by definition. Thus, we can assume that $\mathcal{M}$ is individually rational. In this case, we actually show that every bundled group block of the shape of a generalized tree that arises at $\mathcal{M}$ and is minimal (in the sense of containing the fewest possible number of nodes over all bundled group blocks of the shape of a tree that arise at $\mathcal{M}$) is actually a bundled block.

To show this, suppose that ν is a generalized tree with n nodes and that the bundle-matched types $(i_j,\mathcal{Y}^j)_{1\le j\le n}$ and contract bundles $(z^T)_{T\in\nu}$ comprise a bundled group block of shape ν that arises at $\mathcal{M}$. If the desirability condition in Definition 3′ fails, then for some role $(z,\ell)\in\mathcal{Z}^j$, we must have that $(z,\ell)\notin\mathcal{W}^j\smallsetminus\mathcal{Y}^j$ for some $\mathcal{W}^j\in\mathcal{C}^j(\mathcal{Y}^j\cup\mathcal{Z}^j)$. The preferability condition in Definition B.2.1 entails that $u^{i_j}(\mathcal{W}^j)>u^{i_j}(\mathcal{Y}^j)$. Since the group block arises at $\mathcal{M}$, the individual rationality of $\mathcal{M}$ implies that $\mathcal{Y}^j\in\mathcal{C}^{i_j}(\mathcal{Y}^j)$. Hence, we must have that $\mathcal{W}^j\not\subseteq\mathcal{Y}^j$. In particular, letting $z=z^T$, removing the hyperedge T from ν leads to a graph whose connected component containing j has a nonempty set of edges. Considering this connected component gives rise to a bundled group block of the shape of a generalized tree. (Here, the bundle-matched types and contract bundles in the new group block are defined from the corresponding bundle-matched types and contract bundles in the initial bundled group block.) Since the initial bundled group block arises, so does the new bundled group block. Hence, by considering a minimal bundled group block of the shape of a generalized tree that arises at $\mathcal{M}$, we can conclude that a bundled block of the shape of a generalized tree must arise at $\mathcal{M}$.

C.9. *Proof of Lemma B.2.2*

The key to the proof of the lemma is the following claim.

Claim C.9.1. *Let $(i_1,\mathcal{Y}^1),\dots,(i_n,\mathcal{Y}^n)$ be bundle-matched types. If each contract bundle involving a single contract is feasible and there exists a strict core block (of some shape) whose matched types are $(i_1,\tau(\mathcal{Y}^1)),\dots,(i_n,\tau(\mathcal{Y}^n))$, then there exists a bundled group block whose bundle-matched types are $(i_1,\mathcal{Y}^1),\dots,(i_n,\mathcal{Y}^n)$.*

Proof. Consider a strict core block whose matched types are $(i_1,\tau(\mathcal{Y}^1)),\ldots,(i_n,\tau(\mathcal{Y}^n))$. Suppose that the block has shape ν and that the blocking contracts are $(z_{j,k})_{\{j,k\}\in\nu}$. For each $\{j,k\}\in\nu$, consider a contract bundle $z^{\{j,k\}}$ involving two agents for which $z^{\{j,k\}}_{1,2}=\{z_{j,k}\}$. We claim that the bundle-matched types $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$ and the contract bundles $(z^{\{j,k\}})_{\{j,k\}\in\nu}$ constitute a bundled group block of shape ν. We need to show that the compatibility and group desirability conditions in Definition B.2.1 hold. The compatibility condition in Definition 6 implies the compatibility condition in Definition B.2.1. Letting $\mathcal{Z}^j$ be as in Definition B.2.1, we have that $\tau(\mathcal{Z}^j)=Z^j$ for all $1\le j\le n$ by construction. Due to the definition of u^{i_j}, the preferability condition in Definition 6 implies that the group desirability condition in Definition B.2.1 holds for $\mathcal{W}^j=\mathcal{Z}^j$. ∥

We also need a lemma on equivalence up to bundling. Intuitively, the individual rationality of a bundled outcome entails that no agent (except possibly ones in a set of measure 0) can participate in roles that specify overlapping sets of contracts. Thus, by “unbundling” all contract bundles in an individually rational outcome, we can see that each individually rational bundled outcome is equivalent up to bundling to an outcome (without bundling).

Lemma C.9.1. *If $\mathcal{M}$ is an individually rational bundled outcome, then there exists a unique outcome M that is equivalent to $\mathcal{M}$ up to bundling.*

Proof. Let

$$M^i_Y=\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_i\,|\,\tau(\mathcal{Y})=Y}\mathcal{M}^i_{\mathcal{Y}}$$

for each matched type (i,Y). We first show that $M=(M^i_Y)_{i,Y}$ is an outcome.

The feasibility condition in Definition 1 follows from the feasibility condition in Definition 1′. To show that the reciprocity condition in Definition 1 holds, note that the individual rationality of $\mathcal{M}$ implies that for each bundle-matched type $(i,\mathcal{Y})$ with $\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right)>0$, we have that $u^i(\mathcal{Y})\ge u^i(\varnothing)=u^i(\varnothing)$, and hence that the roles in $\mathcal{Y}$ correspond to pairwise disjoint sets of contracts. Hence, for all bundle-matched types $(i,\mathcal{Y})$ and contracts $x\in\tau(\mathcal{Y})$, we have that

$$\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right)=\sum_{(\chi,\ell)\in\mathcal{Y}\,|\,x\in\chi_\ell}\mu\left(\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right).$$

Therefore, for all contracts $x\in X_{i,j}$, we have that

$$\mu\left(\bigcup_{Y\subseteq X_i\,|\,x\in Y}M^i_Y\right)=\sum_{\mathcal{Y}\subseteq\mathcal{R}_i\,|\,x\in\tau(\mathcal{Y})}\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right)=\sum_{\mathcal{Y}\subseteq\mathcal{R}_i}\sum_{(\chi,\ell)\in\mathcal{Y}\,|\,x\in\chi_\ell}\mu\left(\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right)$$

Exchanging the order of summation yields that

$$\mu\left(\bigcup_{Y\subseteq X_i\,|\,x\in Y}M^i_Y\right)=\sum_{(\chi,\ell)\in\mathcal{R}_i\,|\,x\in\chi_\ell}\left(\sum_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(\ell)}\,|\,(\chi,\ell)\in\mathcal{Y}}\mu\left(\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right)\right)$$

$$=\sum_{(\chi,\ell)\in\mathcal{R}_i\,|\,x\in\chi_\ell}\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(\ell)}\,|\,(\chi,\ell)\in\mathcal{Y}}\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right),$$

where the second equality follows from the feasibility condition in Definition 1′. Expanding the summation then yields that

$$\mu\left(\bigcup_{Y\subseteq X_i\,|\,x\in Y}M^i_Y\right)=\sum_{\text{contract bundles }\chi}\left[\sum_{\ell,\ell'\,|\,\chi(\ell)=i\text{ and }x\in\chi_{\ell,\ell'}}\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(\ell)}\,|\,(\chi,\ell)\in\mathcal{Y}}\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right)\right]$$

Note that if χ is a contract bundle, $x\in\chi_{\ell,\ell'}$, and $\chi(\ell)=i$, then since $x\in X_{i,j}$, Definition 5 requires that $\chi(\ell')=j$. In light of the reciprocity condition in Definition 1′, it follows that

$$\mu\left(\bigcup_{Y\subseteq X_i\,|\,x\in Y}M^i_Y\right)=\sum_{\text{contract bundles }\chi}\left[\sum_{\ell,\ell'\,|\,\chi(\ell)=i\text{ and }x\in\chi_{\ell,\ell'}}\mu\left(\bigcup_{\mathcal{Y}\subseteq\mathcal{R}_{\chi(\ell)}\,|\,(\chi,\ell)\in\mathcal{Y}}\mathcal{M}^{\chi(\ell)}_{\mathcal{Y}}\right)\right]$$

$$= \sum_{\text{contract bundles } \chi} \left[\sum_{\ell,\ell' \mid \chi(\ell')=j \text{ and } x \in \chi_{\ell,\ell'}} \mu \left(\bigcup_{\mathcal{Y} \subseteq \mathcal{R}_{\chi(\ell')} \mid (\chi,\ell') \in \mathcal{Y}} \mathcal{M}_{\mathcal{Y}}^{\chi(\ell')} \right) \right]$$

$$= \mu \left(\bigcup_{Y \subseteq X_j \mid x \in Y} M_Y^j \right)$$

—yielding the reciprocity condition in Definition 1′. Hence, $M = (M_Y^i)_{i,Y}$ is an outcome; furthermore, it is equivalent to $\mathcal{M}$ up to bundling by construction, and is clearly the unique outcome with this property. ‖

To prove Lemma B.2.2, consider a strongly group stable bundled outcome $\mathcal{M}$. Note that since $\mathcal{M}$ is individually rational, Lemma C.9.1 implies that there exists an outcome M that is equivalent to $\mathcal{M}$ up to bundling. We show that M is in the weak core. Suppose for sake of deriving a contradiction that M is not in the weak core. In this case, some strict core block must arise at M—say, one whose matched types are $(i_1,Y^1),\ldots,(i_n,Y^n)$. For each $1 \leq j \leq n$, let $(i_j,\mathcal{Y}^j)$ be a bundle-matched type such that $\mu\left(\mathcal{M}_{\mathcal{Y}^j}^{i_j}\right) > 0$ and $\tau(\mathcal{Y}^j) = Y^j$; such a bundle-matched type exists since M and $\mathcal{M}$ are equivalent up to bundling and $\mu\left(M_{Y^j}^{i_j}\right) > 0$. By Claim C.9.1, there exists a bundled group block whose bundle-matched types are $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$. Such a bundled group block arises at $\mathcal{M}$ by construction—contradicting the hypothesis that $\mathcal{M}$ is strongly group stable. Hence, we can conclude that M is in the weak core—as desired.

C.10. *Proof of Theorem B.2.1*

We prove Parts (a) and (b) separately; the constant N for the overall theorem should be taken to be the larger of the constants from the proofs of the two parts.

Proof of Theorem B.2.1(a). To prove the result, we reformulate the concept of group blocks in a way that relies only on the partitioning of contracts that an agent signs induced by the bundling of these contracts into roles. To construct the constant N, we use the finiteness of the family of partitions of sets of contracts.

Formally, we define a *partitional matched type* $(i,Y,\mathcal{Q})$ to be a matched type (i,Y) together with a partition $\mathcal{Q}$ of Y. Given a bundle-matched type $(i,\mathcal{Y})$ with $\mathcal{Y} \in \mathcal{C}^i(\mathcal{Y})$, the roles in $\mathcal{Y}$ correspond to pairwise disjoint sets of contracts (as $u^i(\mathcal{Z}) < u^i(\varnothing)$ for sets $\mathcal{Z} \subseteq \mathcal{R}$ of roles that do not correspond to pairwise disjoint sets of contracts), and thus there is a corresponding partitional matched type $(i,Y,\mathcal{Q})$ given by $Y = \tau(\mathcal{Y})$ and $\mathcal{Q} = \{\chi_\ell \mid (\chi,\ell) \in \mathcal{Y}\}$. Our proof uses the following version of the concept of (bundled) group blocks that is formulated in terms of partitional matched types.

Definition C.10.1. *Let ν be a graph with n nodes. A* ***partitional group multiblock of shape*** ν *consists of*

- *a partitional matched type $(i_j,Y^j,\mathcal{Q}^j)$ for each node $1 \leq j \leq n$, and*
- *a non-empty set $Z_{j,k} = Z_{k,j} \subseteq X_{i_j,i_k}$ for each edge $\{j,k\} \in \nu$*

for which

- *[compatibility] for each node j, the sets $Z_{j,k}$ are disjoint;*
- *[group desirability] for each node j, writing*

$$Z^j = \bigcup_{\{j,k\} \in \nu} Z_{j,k},$$

there exists a set $\mathcal{R}^j \subseteq \mathcal{Q}^j$ such that $Z^j \cap S = \varnothing$ for all $S \in \mathcal{R}^j$ and writing

$$W^j = Z^j \cup \bigcup_{S \in \mathcal{R}^j} S,$$

we have that $u^{i_j}(W^j) > u^{i_j}(Y^j)$.

The following claim relates partitional group multiblocks to bundled group blocks.

Claim C.10.1. *Let $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$ be bundle-matched types such that $\mathcal{Y}^j\in\mathcal{C}^{i_j}(\mathcal{Y}^j)$ for all $1\le j\le n$. Let $(i_j,Y^j,\mathcal{Q}^j)$ be the partitional matched type corresponding to $(i_j,\mathcal{Y}^j)$.*

(a) *If there exists a bundled group block (of some shape) whose bundle-matched types are $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$, then there exists a partitional group multiblock (of some shape) whose partitional matched types are*
$$(i_1,Y^1,\mathcal{Q}^1),\ldots,(i_n,Y^n,\mathcal{Q}^n).$$
(b) *If all contract bundles involving at most n agents are feasible and there exists a partitional group multiblock (of some shape) whose partitional matched types are*
$$(i_1,Y^1,\mathcal{Q}^1),\ldots,(i_n,Y^n,\mathcal{Q}^n),$$
then there exist $m\le n$, and a sequence $1\le p_1<p_2<\cdots<p_m\le n$, and a bundled group block of shape $\{\{1,2,\ldots,m\}\}$ whose bundle-matched types are $(i_{p_1},\mathcal{Y}^{p_1}),\ldots,(i_{p_m},\mathcal{Y}^{p_m})$.

Proof. We first prove Part (a). Consider a bundled group block of shape ν (where ν is a hypergraph) comprised of bundle-matched types $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$ and contract bundles $(z^T)_{T\in\nu}$. For each $1\le j\le n$, let $\mathcal{W}^j\subseteq\mathcal{Z}^j\cup\mathcal{Y}^j$ be such that $\mathcal{Z}^j\subseteq\mathcal{W}^j$ and $u^{i_j}(\mathcal{W}^j)>u^{i_j}(\mathcal{Y}^j)$. Define a graph ν' with n nodes by
$$\nu'=\left\{\{j,k\}\subseteq\{1,2,\ldots,n\}\,\middle|\,\text{there exists } T\in\nu \text{ with } \{j,k\}\subseteq T \text{ and } z^T_{T^{-1}(j),T^{-1}(k)}\neq\varnothing\right\}.$$
For each $\{j,k\}\in\nu$, define
$$Z_{j,k}=\bigcup_{T\in\nu'\mid\{j,k\}\subseteq T} z^T_{T^{-1}(j),T^{-1}(k)}.$$
By construction, $Z_{j,k}$ is nonempty for all $\{j,k\}\in\nu$.

Since $\mathcal{W}^j$ and $u^{i_j}(\mathcal{W}^j)>u^{i_j}(\mathcal{Y}^j)$ and $\mathcal{Y}^j\in\mathcal{C}^{i_j}(\mathcal{Y}^j)$, the roles in $\mathcal{W}^j$ must consist of pairwise disjoint sets of contracts. Since $\mathcal{Z}^j\subseteq\mathcal{W}^j$, the same must hold for the roles in $\mathcal{Z}^j$. In particular, the sets $(Z_{j,k})_{1\le k\le n}$ must be disjoint for each node j—yielding the compatibility condition in Definition C.10.1. To show that the group desirability condition in Definition C.10.1 holds, we define a set
$$\mathcal{R}^j=\{\chi_\ell\mid(\chi,\ell)\in\mathcal{W}^j\smallsetminus\mathcal{Z}^j\}\subseteq\mathcal{Q}^j.$$
Since the roles in $\mathcal{W}^j$ consist of pairwise disjoint sets of contracts, we have that $Z^j\cap S=\varnothing$ for all $S\in\mathcal{R}^j$. Letting W^j be as in the group desirability condition in Definition C.10.1, we have that
$$W^j=\bigcup_{r\in\mathcal{W}^j}\tau(r),$$
and hence that
$$u^{i_j}(W^j)=u^{i_j}(\mathcal{W}^j)>u^{i_j}(\mathcal{Y}^j)=u^{i_j}(Y^j),$$
yielding the group desirability condition in Definition C.10.1. Hence, the partitional matched types $(i_1,Y^1,\mathcal{Q}^1),\ldots,(i_n,Y^n,\mathcal{Q}^n)$ and the sets $(Z_{j,k})_{\{j,k\}\in\nu'}$ of contracts comprise a partitional group multiblock of shape ν'.

We next prove Part (b). Consider a partitional group multiblock of shape ν' comprised of the partitional matched types $(i_1,Y^1,\mathcal{Q}^1),\ldots,(i_n,Y^n,\mathcal{Q}^n)$ and the sets $(Z_{j,k})_{\{j,k\}\in\nu'}$ of contracts. First, we suppose that ν' is connected. In this case, we can define a contract bundle z involving n agents by $z(j)=i_j$ for $1\le j\le n$ and
$$z_{j,k}=\begin{cases}Z_{j,k} & \text{if } \{j,k\}\in\nu'\\ \varnothing & \text{if } \{j,k\}\notin\nu'\end{cases}.$$
The compatibility condition in Definition C.10.1 implies the compatibility condition in Definition 5. Considering the hypergraph $\nu=\{\{1,2,\ldots,n\}\}$, we let $z^T=z$ for $T=\{1,2,\ldots,n\}$. The compatibility condition in Definition B.2.1 clearly holds. To show that the group desirability condition in Definition B.2.1 holds, let $\mathcal{R}^j\subseteq\mathcal{Q}^j$ be such that $u^{i_j}(W^j)>u^{i_j}(Y^j)$ in the group desirability condition in Definition C.10.1. Define $\mathcal{W}^j\subseteq\mathcal{Z}^j$ by
$$\mathcal{W}^j=\{(z,j)\}\cup\{(\chi,\ell)\in\mathcal{Y}^j\mid\chi_\ell\in\mathcal{R}^j\}.$$
Since $\mathcal{Y}^j\in\mathcal{C}^{i_j}(\mathcal{Y}^j)$ and $Z^j\cap S=\varnothing$ for all $S\in\mathcal{R}^j$, the roles in $\mathcal{W}^j$ consist of pairwise disjoint sets of contracts. Moreover, we have that $\tau(\mathcal{W}^j)=W^j$ by construction. It follows that
$$u^{i_j}(\mathcal{W}^j)=u^{i_j}(W^j)>u^{i_j}(Y^j)=u^{i_j}(\mathcal{Y}^j)$$
—yielding the group desirability condition in Definition B.2.1. Hence, the bundle-matched types $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$ and the contract $z=z^T$ comprise a bundled group block of shape $\nu=\{\{1,2,\ldots,n\}\}$.

Now, if ν' is not connected, we can apply the same argument after passing to a connected component to prove Part ((b)) of the claim. $\quad\|$

To complete the proof of the proposition, we first construct the constant N for a given set X of contracts. Fix a profile $\succ = (\succ_i)_{i\in I}$ of preferences over sets of contracts. Consider the family $\mathcal{F}(\succ)$ of sets S of partitional matched types such that when type i's utility function represents $\succ_i$, we have that $S\in\mathcal{F}(\succ)$ if and only if there is a partitional group multiblock involving only partitional matched types in S. For each $S\in\mathcal{F}(\succ)$, fix a partitional group multiblock of the shape of a graph $\nu(S,\succ)$ with $n(S,\succ)$ nodes that consists of partitional matched types $(i_j(S,\succ),Y^j(S,\succ),\mathcal{Q}^j(S,\succ))\in S$ and sets $Z_{i,j}(S,\succ)\subseteq X_{i_j,i_k}$ of contracts for $\{i,j\}\in\nu$. Let

$$N=\max_{\text{profiles }\succ}\max_{S\in\mathcal{F}(\succ)} n(S,\succ).$$

Now, suppose that every contract bundle involving at most N agents is feasible. We prove the contrapositive of the assertion of the proposition: i.e., that every bundled outcome that is not strongly group stable is not tree-stable. Consider a bundled outcome $\mathcal{M}$ that is not strongly group stable. If $\mathcal{M}$ is not individually rational, then it is not tree-stable. Hence, we assume that $\mathcal{M}$ is individually rational and suppose that a bundled group block of some shape arises. Let $\mathcal{S}$ denote the set of bundle-matched types involved in the bundled group block; we have that $\mu(\mathcal{M}^i_{\mathcal{Y}})>0$ for each $(i,\mathcal{Y})\in\mathcal{S}$ by construction. Let S denote the set of partitional matched matched types corresponding to the bundle-matched types in $\mathcal{S}$. Letting $\succ$ denote the profile of agents' preferences over sets of contracts, Claim C.10.1(a) implies that $S\in\mathcal{F}(\succ)$. As $n(S,\succ)\le N$, Claim C.10.1(b) then implies that there exist $m\le n(S,\succ)$ and a bundled group block of shape $\{\{1,2,\dots,m\}\}$ whose bundle-matched types are elements of $\mathcal{S}$. Such a bundled group block arises at $\mathcal{M}$ by construction. As $\{\{1,2,\dots,m\}\}$ is a generalized tree, Proposition B.2.1 implies that $\mathcal{M}$ cannot be tree-stable.

Proof of Theorem B.2.1(b). We first construct the constant N for a given set X of contracts. Consider the set

$$D'=\left\{ d\in \bigtimes_{i\in I}\mathbb{R}_{\ge 0}^{\mathcal{P}(X_i)\smallsetminus\{\varnothing\}} \;\middle|\; \sum_{Y\in\mathcal{P}(X_i)\mid x\in Y} d^i_Y=\sum_{Y\in\mathcal{P}(X_j)\mid x\in Y} d^j_Y \text{ for all } x\in X_{i,j}\right\}.$$

As X is finite, D' is a rational polyhedral cone. By the Weyl–Minkowski Theorem, D' is generated by a finite set of integer vectors: say, $\{d(1),d(2),\dots,d(Q)\}$. By the Krein–Millman Theorem for cones, we can take $d(1),\dots,d(Q)$ to be extreme rays of D'. Without loss of generality, we can assume that each vector $d(q)$ is primitive in the sense that the greatest common divisor of its components is 1. These reductions ensure that no vector $d(q)$ can be expressed as the sum of two nonzero integral elements of D'.

For each $1\le q\le Q$, let

$$n_q=\sum_{i\in I}\sum_{Y\in\mathcal{P}(X_i)\smallsetminus\{\varnothing\}} d^i_Y(q).$$

Furthermore, let

$$N=\max_{q\in\{1,\dots,Q\}} n_q.$$

Now, suppose that all contract bundles involving at most N agents are feasible. In particular, for each $1\le q\le Q$, all contract bundles involving n_q agents are feasible. For each $1\le q\le Q$, consider a contract bundle $\chi^q\in\mathcal{X}[n_q]$ for which, writing

$$L^i_Y(q)=\{1\le\ell\le n_q\mid \chi^q(\ell)=i \text{ and } \chi_\ell=Y\}$$

for the set of indices of roles that prescribe the set Y of contracts, we have that

$$|L^i_Y(q)|=\begin{cases} d^i_Y(q) & \text{if } Y\neq\varnothing\\ 0 & \text{if } Y=\varnothing\end{cases} \tag{C.1}$$

for each matched type (i,Y). Such contract bundles exist because $d(1),\dots,d(q)$ are integral elements of D'. (Here, the connectness of χ^q follows from the fact that $d(q)$ cannot be expressed as a sum of two nonzero integral elements of D'.)

We need to show that each strict core outcome is equivalent up to bundling to a strongly group stable outcome. Consider a strict core outcome $M=(M^i_Y)_{i,Y}$. For each matched type (i,Y) with $Y\neq\varnothing$, let

$$d^i_Y=\mu\left(M^i_Y\right).$$

The reciprocity condition in Definition 1 implies that $d \in D'$. As $d(1),\ldots,d(Q)$ generate D' as a cone, we can write

$$d = \sum_{q=1}^{Q} \lambda_q d(q). \tag{C.2}$$

for some $\lambda_1,\ldots,\lambda_Q \geq 0$.

We now construct a bundled outcome that is equivalent up to bundling to the strict core outcome M, and show that it is strongly group stable. Consider a bundled outcome $\mathcal{M}$ with the properties that

- we have that $\mathcal{M}^i_{\mathcal{Y}} = \varnothing$ for $|\mathcal{Y}| > 1$, and for $\mathcal{Y} = \{(\chi,\ell)\}$ with $\chi \notin \{\chi^1,\ldots,\chi^Q\}$;
- for all indices $1 \leq q \leq Q$ and $1 \leq \ell \leq n_q$, we have that $\mu\left(\mathcal{M}^{\chi^q(\ell)}_{(\chi^q,\ell)}\right) = \lambda_q$; and
- for all matched types (i,Y), we have that

$$M^i_Y = \bigcup_{1 \leq q \leq Q} \bigcup_{\ell \in L^i_Y(q)} \mathcal{M}^i_{(\chi^q,l)}$$

Thus, in $\mathcal{M}$, every agent takes at most one role, and agents only take roles in contract bundles $\chi^1,\ldots,\chi^Q$. Such a bundled outcome exists due to (C.1) and (C.2). By construction, $\mathcal{M}$ is equivalent up to bundling to M.

It remains to show that $\mathcal{M}$ is strongly group stable. We first show that $\mathcal{M}$ is individually rational. Consider a bundle-matched type $(i,\mathcal{Y})$ with $\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right) > 0$. Since M is in the strict core, the matched type $(i,\tau(\mathcal{Y}))$ cannot define a weak core block of shape $\nu = \varnothing$. Hence, we must have that $u^i(\tau(\mathcal{Y})) \geq u^i(\varnothing)$. As $\mathcal{Y}$ consists of a single role, we have that

$$u^i(\mathcal{Y}) = u^i(\tau(\mathcal{Y})) \geq u^i(\varnothing) = u^i(\varnothing),$$

and hence that $\mathcal{Y} \in \mathcal{C}^i(\mathcal{Y})$.

To show that no bundled group block arises at $\mathcal{M}$, we consider the following weakening of bundled group blocks.

Definition C.10.2. *Let ν be a hypergraph with n nodes. A **weak bundled group block of shape** ν consists of*

- *a bundle-matched type $(i_j,\mathcal{Y}^j)$ for each node $1 \leq j \leq n$,*
- *a contract bundle $z^T \in \mathcal{X}[|T|]$ for each hyperedge $T \in \nu$ with $z^T(\ell) = i_{T(\ell)}$ for all indices $1 \leq \ell \leq |T|$, and*
- *for each node j, writing*
$$\mathcal{Z}^j = \{(z^T,\ell) \mid T \in \nu \text{ and } T(\ell) = j\}$$
for the set of roles in blocking contracts assigned to the node j, a set $\mathcal{W}^j \subseteq \mathcal{Y}^j \cup \mathcal{Z}^j$ with $\mathcal{Z}^j \subseteq \mathcal{W}^j$

for which

- *[compatibility] for each node j, the roles (z^T,ℓ) with $T(\ell) = j$ are distinct;*
- *[weak group desirability] we have that $u^{i_j}(\mathcal{W}^j) \geq u^{i_j}(\mathcal{Y}^j)$ for all $1 \leq j \leq n$, with strict inequality for some j.*

*Such a weak bundled group block is **core-like** if $\mathcal{W}^j = \mathcal{Z}^j$ for all $1 \leq j \leq n$. Such a weak bundled group block **arises** at a bundled outcome $\mathcal{M}$ if $\mu\left(\mathcal{M}^{i_j}_{\mathcal{Y}^j}\right) > 0$ for all $1 \leq j \leq n$.*

The following claim relates weak bundled group blocks to weak core blocks.

Claim C.10.2. *Let $\mathcal{M}$ be a bundled outcome.*

(a) If $\mu\left(\mathcal{M}^i_{\mathcal{Y}}\right) = 0$ for all bundle-matched types $(i,\mathcal{Y})$ with $|\mathcal{Y}| > 1$ and a weak bundled group block (of some shape) arises at $\mathcal{M}$, then a core-like weak bundled group block (of some shape) arises at $\mathcal{M}$.

(b) If $\mathcal{M}$ is individually rational, a core-like weak bundled group block (of some shape) arises at $\mathcal{M}$, and M is an outcome that is equivalent to $\mathcal{M}$ up to bundling, then a weak core block (of some shape) arises at M.

Proof. We first prove Part (a). Among all weak bundled group blocks that arise at $\mathcal{M}$, consider one that minimizes the number of indices j for which $\mathcal{W}^j \not\subseteq \mathcal{Z}^j$. Suppose that it is of shape ν and consists of bundle-matched types $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$, contract bundles $(z^T)_{T\in\nu}$, and sets $\mathcal{W}^j \subseteq \mathcal{Y}^j \cup \mathcal{Z}^j$ of roles. We show that $\mathcal{W}^j = \mathcal{Z}^j$ for all $1\le j\le n$. Suppose for sake of deriving a contradiction that $\mathcal{W}^k \not\subseteq \mathcal{Z}^k$. As the weak bundled group block arises, we must have that $\mathcal{W}^k = \{(\chi,\ell^*)\}\cup \mathcal{Z}^k$, where $\chi\in X[N]$ and $1\le \ell^*\le N$.

Define a function $\pi:\mathcal{P}(\{1,2,\ldots,n\})\to\mathcal{P}(\{1,2,\ldots,n+N-1\})$ by

$$\pi(T)=\{j+\ell^*-1 \mid j\in T\}.$$

Consider the set $T^0=\{\{1,2,\ldots,\ell^*-1,n+\ell^*,n+\ell^*+1,\ldots,n+N-1\}\}$ and the hypergraph

$$\nu'=\{T^0\}\cup\{\pi(T)\mid T\in\nu\}$$

with $n+N-1$ nodes; we construct a weak bundled group block of shape ν' as follows.

- The bundle-matched types are $(i'_1,\mathcal{Y}'^{,1}),\ldots,(i'_{n+N-1},\mathcal{Y}'^{,n+N-1})$, where

$$(i'_j,\mathcal{Y}'^{,j})=\begin{cases}(\chi(j),\{(\chi,j)\}) & \text{if } 1\le j\le \ell^*-1\\ \left(i_{j-\ell^*+1},\mathcal{Y}^{j-\ell^*+1}\right) & \text{if } \ell^*\le j\le n+\ell^*-1\\ (\chi(j-n+1),\{(\chi,j-n+1)\}) & \text{if } n+\ell^*\le j\le n+N-1\end{cases}.$$

- The contract bundles are given by $(z'^{,T})_{T\in\nu'}$, where we let

$$z'^{,T'}=\begin{cases} z^{T'} & \text{if } T'=\pi(T),\text{ where } T\in\nu\\ \chi & \text{if } T'=T^0\end{cases}.$$

  Hence, the set of roles in new contracts assigned to the jth participating agent is

$$\mathcal{Z}'^{,j}=\begin{cases}\mathcal{Y}'^{,j} & \text{if } 1\le j\le\ell^*-1 \text{ or } n+\ell^*-1\le j\le n+N-1\\ \mathcal{Z}^{j-\ell^*+1} & \text{if } \ell^*\le j\le n+\ell^*-1 \text{ and } j\ne \ell^*+k-1\\ \mathcal{Y}^k\cup\mathcal{Z}^k & \text{if } j=\ell^*+k-1\end{cases}.$$

- The sets of roles are given by

$$\mathcal{W}'^{,j}=\begin{cases}\mathcal{W}^{j-\ell^*+1} & \text{if } \ell^*\le j\le n+\ell^*-1 \text{ and } j\ne\ell^*+k-1\\ \mathcal{Z}'^{,j} & \text{otherwise}\end{cases}.$$

The compatibility and weak group desirability conditions are consequences of the conditions for the initial weak bundled group block, respectively. The reciprocity condition in Definition 1′ implies that $\mu\left(\mathcal{M}^{i'_j}_{\mathcal{Y}'^{,j}}\right)=\mu\left(\mathcal{M}^{i_k}_{\mathcal{Y}^k}\right)$ for $1\le j\le\ell^*-1$ and $n+\ell^*\le j\le n+N-1$. Thus, the hypothesis that the initial weak bundled group block arises implies that the constructed one does as well. However, we can easily see that the number of indices j for which $\mathcal{W}'^{,j}\not\subseteq\mathcal{Z}'^{,j}$ is smaller than in the initial weak bundled group block—contradicting the choice of the initial weak group bundled block. Thus, we can conclude that the initial weak bundled group block is core-like, so in particular, a core-like weak bundled group block (of some shape) arises.

We next prove Part (b). Let M be an outcome equivalent up to bundling to $\mathcal{M}$. Consider a core-like weak bundled group block of shape ν that arises—say of shape ν and consisting of bundle-matched types $(i_1,\mathcal{Y}^1),\ldots,(i_n,\mathcal{Y}^n)$, contract bundles $(z^T)_{T\in\nu}$, and sets $\mathcal{W}^j=\mathcal{Z}^j$ of contracts. As $\mathcal{M}$ is individually rational, we have that $u^{i_j}(\mathcal{Y}^j)\ge u^{i_j}(\varnothing)$ for all $1\le j\le n$.

Define a graph ν' with n nodes by

$$\nu'=\left\{\{j,k\}\subseteq\{1,2,\ldots,n\}\,\middle|\,\text{there exists } T\in\nu \text{ with } \{j,k\}\subseteq T \text{ and } z^T_{T^{-1}(j),T^{-1}(k)}\ne\varnothing\right\},$$

and let

$$Z_{j,k}=\bigcup_{T\in\nu\mid\{j,k\}\subseteq T} z^T_{T^{-1}(j),T^{-1}(k)}$$

for $\{j,k\}\in\nu'$. By construction, $Z_{j,k}$ is nonempty for all $\{j,k\}\in\nu$. The weak group desirability condition in Definition C.10.2 requires that $u^{i_j}(\mathcal{W}^j)\ge u^{i_j}(\mathcal{Y}^j)$; since $\mathcal{W}^j=\mathcal{Z}^j$ and $u^{i_j}(\mathcal{Y}^j)\ge u^{i_j}(\varnothing)$, we have that $u^{i_j}(\mathcal{Z}^j)\ge u^{i_j}(\varnothing)$. In particular, the roles in $\mathcal{Z}^j$ must therefore correspond to pairwise disjoint sets of contracts, so the sets $(Z_{j,k})_{\{j,k\}\in\nu}$ must be pairwise disjoint for each fixed node j. Letting Z^j be as in Definition 6, we have that $\tau(\mathcal{Z}^j)=Z^j$, and hence that $u^{i_j}(Z^j)\ge u^{i_j}(\mathcal{Z}^j)$. Since $u^{i_j}(\mathcal{Y}^j)\ge u^{i_j}(\varnothing)$, we must have that $u^{i_j}(\tau(\mathcal{Y}^j)) = u^{i_j}(\mathcal{Y}^j)$. Hence, the weak group desirability condition in Definition C.10.2 implies that

$$u^{i_j}(Z^j) \geq u^{i_j}(\mathcal{Z}^j) = u^{i_j}(\mathcal{W}^j) \geq u^{i_j}(\mathcal{Y}^j) = u^{i_j}(\tau(\mathcal{Y}^j))$$

for all $1 \leq j \leq n$ with strict inequality for some j—yielding the preferability condition for weak core multiblocks in Definition B.1.2.

Thus, the matched types $(i_1, \tau(\mathcal{Y}^1)), \ldots, (i_n, \tau(\mathcal{Y}^n))$ and the sets $(Z_{j,k})_{\{j,k\} \in \nu'}$ of contracts comprise a weak core multiblock of shape ν'. As M and $\mathcal{M}$ are equivalent up to bundling, this weak core multiblock arises. By Proposition B.1.2, a weak core block (of some shape) must arise as well. ‖

Since M is in the strict core, the contrapositive of Claim C.10.2(b) implies that no core-like weak bundled group block (of any shape) can arise at $\mathcal{M}$. The contrapositive of Claim C.10.2(a) then implies that no weak bundled group block (of any shape) can arise at $\mathcal{M}$. But as each bundled group block that arises at $\mathcal{M}$ clearly gives rise to a weak bundled group block that arises at $\mathcal{M}$, it follows that no bundled group block of any shape can arise at $\mathcal{M}$. Hence, $\mathcal{M}$ is strongly group stable—as desired.

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